Supersymmetric valence bond solid states

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In this work we investigate the supersymmetric version of the valence bond solid (SVBS) state. In one dimension, the SVBS states continuously interpolate between the valence bond states for integer and half-integer spin chains, and they generally describe superconducting valence bond liquid states. Spin and superconducting correlation functions can be computed exactly for these states and their correlation lengths are equal at the supersymmetric point. In higher dimensions, the wave function for the SVBS states can describe resonating valence bond states. The SVBS states for the spin models are shown to be precisely analogous to the bosonic Pfaffian states of the quantum Hall effect. We also give microscopic Hamiltonians for which the SVBS state is the exact ground state.

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I. INTRODUCTION

Quantum antiferromagnetism offers basic paradigms for different phases of strongly interacting quantum systems.[1–3] In addition to a rich array of classically ordered states, including multiple sublattice Néel order and noncollinear states, there are several different types of quantum disordered states: valence bond (VB) solids, valence bond liquids, dimer solids, etc. By tuning various couplings, one can pass through quantum phase transitions which separate these states. A class of superconductors, including the high-T_c cuprates, is obtained by doping the Mott insulating states with quantum antiferromagnetic order. In one theoretical approach, superconductivity arises from doping the valence bond liquid state.[4] In another theoretical approach, the superconducting state is obtained from a symmetry rotation of the quantum antiferromagnetic state.[5] In this work we construct supersymmetric extension of the valence bond solid state. In particular, we show that the superconducting valence bond liquid state can be naturally obtained from the supersymmetric rotation of the valence bond solid state. Our results give a mathematical precise validation of the above-mentioned ideas.

We investigate extensions of the valence bond solid states defined by Affleck, Kennedy, Lieb, and Tasaki (AKLT).[6,7] On any lattice $\mathcal{L}$, one can define a one-parameter family of such states, indexed by an integer M. The general AKLT state is written[8]

$$|\Phi(\mathcal{L},M)\rangle = \prod_{\langle ij\rangle} (\epsilon_{\mu\nu} b^\dagger_{i\mu} b^\dagger_{j\nu})^M |0\rangle, \tag{1}$$

as a product over links $\langle ij\rangle$ of $\mathcal{L}$, where $\boldsymbol{S}_i = \frac{1}{2} b^\dagger_{i\mu} \boldsymbol{\sigma}_{\mu\nu} b_{i\nu}$ is the local quantum spin operator, written in terms of Schwinger bosons, satisfying $[b_{i\mu}, b^\dagger_{j\nu}] = \delta_{ij}\delta_{\mu\nu}$. The state $|\Phi(\mathcal{L},M)\rangle$ describes an antiferromagnet where each site contains a single spin $S = \frac{1}{2}zM$ object, where z is the lattice coordination number. What is special about these states is that the total spin J on any link can only take values between 0 and $J^* \equiv (z-1)M$, and all total spin components $J = J^*+1, \ldots, 2S$ along any link are absent in the AKLT wave function because the operator $\phi^\dagger_{ij} = \epsilon_{\mu\nu} b^\dagger_{i\mu} b^\dagger_{j\nu}$ transforms as an SU(2) singlet. Thus, the AKLT states are annihilated by certain projection operators,

$$\mathrm{P}_J(ij)\,|\Phi(\mathcal{L},M)\rangle = 0 \tag{2}$$

for $J \in [J^*+1, 2S]$, where $J^* = (z-1)M$. This allows one to construct local Hamiltonians of the form

$$H = \sum_{\langle ij\rangle} \sum_{J^*+1}^{2S} V_J \mathrm{P}_J(ij) \tag{3}$$

with $V_J \geq 0$, which renders $|\Phi(\mathcal{L},M)\rangle$ an exact, zero-energy ground state. In this respect, the AKLT states are analogous to the Laughlin wave functions in the fractional quantum Hall effect (QHE),[9] which are also rendered exact eigenstates of a corresponding "truncated pseudopotential" Hamiltonian.[8,10] The SU(2) AKLT model has been generalized by introducing q-deformed SU(2) group[11–13] and higher symmetric groups, such as SU(n),[14–17] SP(n),[18] and SO(n).[19,20]

The states we shall discuss are supersymmetric generalizations of the AKLT states and are written as

$$|\Psi(\mathcal{L},M,r)\rangle = \prod_{\langle ij\rangle} (\epsilon_{\mu\nu} b^\dagger_{i\mu} b^\dagger_{j\nu} + r f^\dagger_i f^\dagger_j)^M |0\rangle. \tag{4}$$

Here, $f^\dagger_i$ creates a fermionic hole on site i, which displaces one of the bosons. The local Hilbert space thus accommodates two types of states: states with spin $S = \frac{1}{2}zM$ and states with spin $S - \frac{1}{2}$, and the operator $\chi^\dagger_{ij} = \epsilon_{\mu\nu} b^\dagger_{i\mu} b^\dagger_{j\nu} + r f^\dagger_i f^\dagger_j$, which creates a linear combination of spin singlets and hole pairs on the link $\langle ij\rangle$, transforms as a singlet under the superalgebra OSp(1|2). We call these states supersymmetric valence bond solid (SVBS) states. Physically, the spin S states can be realized by $2S$ electrons coupled through Hund's rule cou-

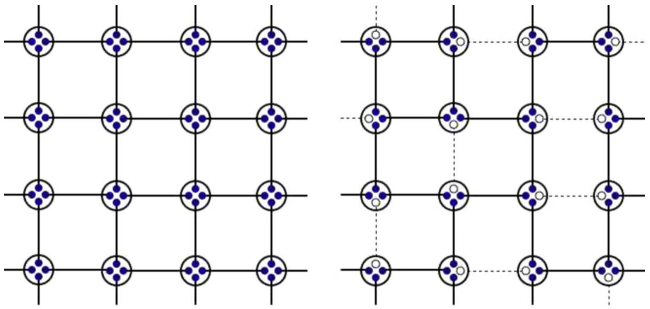

FIG. 1. (Color online) Examples of the square lattice supersymmetric valence bond solid state with $M=1$ on the square lattice. Left panel: $r=0$, corresponding to the $S=2$ AKLT state. Right panel: example configuration from the $r=\infty$ state which is a $S=\frac{3}{2}$ nearest-neighbor RVB state.

pling and the spin $S-\frac{1}{2}$ states are obtained by removing one electron from the site. Thus the SVBS states describe a doped spin chain with large on-site Hund's rule coupling.

The parameter r interpolates between two limits. At $r=0$, there are no holes and we recover the AKLT state, which is an antiferromagnetic insulator. For finite r, there is a nonzero density of nearest-neighbor hole pairs and the system is a superconductor. The average spin per site is somewhere between $S-\frac{1}{2}$ and S. As $r\to\infty$, each site must contain a hole and the state is insulating once again. For a one-dimensional (1D) chain, with $M=1$, there are only two possibilities:

$$|\Phi_{\mathrm{A}}\rangle = |\bullet\!\!-\!\!\bullet\;\bullet\!\!-\!\!\bullet\;\bullet\!\!-\!\!\bullet\;\bullet\!\!-\!\!\bullet\;\bullet\rangle \tag{5a}$$

$$|\Phi_{\mathrm{B}}\rangle = |\;\bullet\!\!-\!\!\bullet\;\bullet\!\!-\!\!\bullet\;\bullet\!\!-\!\!\bullet\;\bullet\!\!-\!\!\bullet\;\rangle, \tag{5b}$$

corresponding to spin-Peierls order. These are the two degenerate ground states of the well-known Majumdar-Ghosh Hamiltonian.[21–23] In the thermodynamic limit, or on a ring with an even number of sites, the $r\to\infty$ SVBS state is the sum $|\Phi_A\rangle+|\Phi_B\rangle$, which has crystal momentum $k=0$.

On the two-dimensional square lattice, once again the $r=0$ state is the $S=2$ AKLT valence bond solid. For $r\to\infty$, though, rather than there being only two configurations which contribute to the SVBS wave function, the state is a linear combination of the resonating valence bond (RVB) kind but for $S=\frac{3}{2}$. The situation is depicted in Fig. 1. The configurations which contribute to the SVBS state in this limit are given by dimer coverings of the square lattice, where each dimer corresponds to a hole-pair-creation operator $f_i^\dagger f_j^\dagger$. The quantum dimer model for $S=\frac{1}{2}$ was constructed by Rokhsar and Kivelson.[24] The partition function for the classical dimer gas, with different fugacities for x-directed and y-directed dimers, was worked out by Fisher[25] in 1961 and shown to take the form of a Pfaffian. This connection to the Pfaffian is present in our work as well, and underlies recent work by one of us[26–28] on supersymmetric extensions of the quantum Hall problem, in which the Pfaffian QHE state at $\nu=\frac{1}{2}$ appears as a natural limit. The following diagram sketches these basic connections:

$$\begin{array}{ccc}
\mathrm{AKLT}(S=1\ \ \mathrm{chain}) & \to & \mathrm{Majumdar\text{-}Ghosh}\left(S=\frac{1}{2}\right) \\
\downarrow & & \downarrow \\
\mathrm{AKLT}(S=2\ \ \mathrm{square}) & \to & \mathrm{RVB}\left(S=\frac{3}{2}\right) \\
\downarrow & & \downarrow \\
\mathrm{Laughlin}\left(\nu=\frac{1}{2}\mathrm{bosons}\right) & \to & \mathrm{Pfaffian}\left(\nu=\frac{1}{2}\mathrm{fermions}\right).
\end{array}$$

While the literature on hole motion in quantum antiferromagnets is voluminous rather little has been done to date to explore extensions of models of the AKLT type, i.e., to find wave functions at finite hole concentration which are exact eigenstates of local projectors (so-called "Klein models"[29]). Single hole motion in the $S=1$ AKLT chain was discussed in Ref. 30 but a different constraint was used: $b_\mu^\dagger b_\mu+2f^\dagger f=2$. Experiments on hole doped AKLT spin chains have been reported in Ref. 31, where a hole divides the $S=1$ AKLT chain to two semi-indefinite segments with $S=\frac{1}{2}$ spins at each edge. Interestingly, such property is shared in the SVBS model developed in this paper.

The t-J models with $\mathsf{SU}(2|1)$ symmetry are known as the supersymmetric t-J models. The models are exactly solvable in one dimension[32–34] and their correlation functions are also derived in Ref. 35. With $1/r^2$ long-range interaction, the supersymmetric t-J models are still exactly solvable.[36] The models which we deal with possess $\mathsf{OSp}(1|2)$ symmetry and their exact ground states are constructed even in higher dimensions as the case of the original AKLT models.

The remainder of this paper is structured as follows. In Sec. II we will briefly discuss the local Hilbert space and some preliminary aspects of the $\mathsf{OSp}(1|2)$ operator algebra, a fuller treatment of which we consign to Appendix A. In Sec. III we will focus on SVBS states in one-dimensional systems, i.e., supersymmetric spin chains. Using the spin-hole coherent states developed by Auerbach,[37] we will compute various correlation functions in the SVBS chains. Section IV discusses some connections with the quantum Hall effect. In Sec. V, we derive a Hamiltonian with local interactions which renders our $M=1$ SVBS chain as an exact nondegenerate ground state.

II. LOCAL HILBERT SPACE AND $\mathsf{OSp}(1|2)$

In the Schwinger representation of $\mathsf{SU}(2)$, a spin is represented by two bosons with the quantum spin operator given by $\boldsymbol{S}=\frac{1}{2}b_\mu^\dagger\boldsymbol{\sigma}_{\mu\nu}b_\nu$. The total boson occupancy is constrained,

$$a^\dagger a + b^\dagger b = p, \tag{6}$$

where p is an integer and where we define $b_\uparrow\equiv a$ and $b_\downarrow\equiv b$. The integer p determines the representation of $\mathsf{SU}(2)$; it corresponds to the number of columns in the corresponding Young diagram. The spin magnitude is simply $S=\frac{1}{2}p$ and the dimension of the representation is $g=p+1$.

Now let us add in hole states. The constraint equation becomes

$$a^\dagger a + b^\dagger b + f^\dagger f = p. \tag{7}$$

There are now $g=2p+1$ possible states, corresponding to the two classes

$$S = \frac{1}{2}p \quad : \quad a^\dagger a + b^\dagger b = p \quad \text{and} \quad f^\dagger f = 0,$$

$$S = \frac{1}{2}(p-1) \quad : \quad a^\dagger a + b^\dagger b = p-1 \quad \text{and} \quad f^\dagger f = 1.$$

The simplest such case we shall deal with is $p=2$, for which $g=5$. Explicitly, these states are given by

$$|+1\rangle = \frac{1}{\sqrt{2}}(a^\dagger)^2|\mathsf{V}\rangle, \quad \left|+\frac{1}{2}\right\rangle = f^\dagger a^\dagger|\mathsf{V}\rangle,$$

$$|0\rangle = a^\dagger b^\dagger|\mathsf{V}\rangle,$$

$$|-1\rangle = \frac{1}{\sqrt{2}}(b^\dagger)^2|\mathsf{V}\rangle, \quad \left|-\frac{1}{2}\right\rangle = f^\dagger b^\dagger|\mathsf{V}\rangle,$$

where $|\mathsf{V}\rangle$ is the vacuum for bosons and fermions: $a|\mathsf{V}\rangle = b|\mathsf{V}\rangle = f|\mathsf{V}\rangle = 0$.

The $2p+1$ states obeying the constraint of Eq. (7) may be grouped into a multiplet of the superalgebra $\mathsf{OSp}(1|2)$. This group has five generators, three of which are the familiar $\mathsf{SU}(2)$ spin operators: $L_a = \frac{1}{2}b_\mu^\dagger \sigma^a_{\mu\nu} b_\nu$ with $i=1,2,3$. The remaining two generators are non-Hermitian and may be taken to be

$$K_1 = \frac{1}{2}(x^{-1} f a^\dagger + x f^\dagger b),$$

$$K_2 = \frac{1}{2}(x^{-1} f b^\dagger - x f^\dagger a), \tag{8}$$

where x is an arbitrary complex number. The relations among the generators are

$$[L_a, L_b] = i\epsilon_{abc} L_c,$$

$$[L_a, K_\mu] = \frac{1}{2}\sigma^a_{\nu\mu} K_\nu,$$

$$\{K_\mu, K_\nu\} = \frac{1}{2}(i\sigma^y \sigma^a)_{\mu\nu} L_a. \tag{9}$$

The algebra of the generators is independent of the parameter x. Note that $K_1^2 = \frac{1}{4}a^\dagger b = \frac{1}{4}L_+$, so K_1 is like the "square root" of the angular momentum raising operator $L_+ = L_1 + iL_2$. Since $[L_3, K_1] = \frac{1}{2}K_1$, we have that K_1 raises L_3 by half. Similarly, K_2 lowers L_3 by half and functions as the square root of the angular-momentum lowering operator L_-.

The Casimir operator is given by

$$\mathcal{C} = \boldsymbol{L}^2 + \epsilon_{\mu\nu} K_\mu K_\nu. \tag{10}$$

Acting on the single-site states defined above, $\mathcal{C}$ takes the value $\frac{1}{4}p(p+1)$. Generally, one has $\mathcal{C} = L(L+\frac{1}{2})$, where L, which is either integer or half odd integer, is the maximum eigenvalue of L_3, for a given value of $\mathcal{C}$. We call this quantity L the angular momentum. The dimension of the representation with angular momentum L is $g=4L+1$. The addition of angular momenta within $\mathsf{OSp}(1|2)$ is similar to the $\mathsf{SU}(2)$ case, except the spacing between consecutive L values is $\frac{1}{2}$ rather than 1,

$$L \otimes L' = |L-L'| \oplus \left(|L-L'| + \frac{1}{2}\right) \oplus \cdots \oplus (L+L'). \tag{11}$$

For example, if the local representation of $\mathsf{OSp}(1|2)$ on a single site is $L=1$, then on any link (ij), one can have

$$1 \otimes 1 = 0 \oplus \frac{1}{2} \oplus 1 \oplus \frac{3}{2} \oplus 2, \tag{12}$$

where the dimensions of the five irreducible representations in the product are 1, 3, 5, 7, and 9. The Casimir operator for the two-site system is

$$\mathcal{C}(ij) = 2\boldsymbol{L}(i)\cdot\boldsymbol{L}(j) + \frac{1}{2}F_{ij}^\dagger f_j^\dagger f_i + \frac{1}{2}F_{ij} f_i^\dagger f_j + \frac{1}{2}x^{-2}A_{ij}^\dagger f_i f_j$$

$$+ \frac{1}{2}x^2 A_{ij} f_i^\dagger f_j^\dagger + \mathcal{C}(i) + \mathcal{C}(j), \tag{13}$$

where

$$A_{ij} = a_i b_j - b_i a_j, \tag{14a}$$

$$F_{ij} = a_i^\dagger a_j + b_i^\dagger b_j. \tag{14b}$$

Using the Casimir operator, we can construct projection operators onto representations of a desired value of L. This can be used to construct a Hamiltonian along the lines of AKLT; this program is carried out in Sec. V A. The link operator

$$\chi_{ij}^\dagger = a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger + r f_i^\dagger f_j^\dagger \tag{15}$$

transforms as an $\mathsf{OSp}(1|2)$ singlet whenever $x^2 = -r$. That is to say $[L_a, \chi_{ij}^\dagger] = [K_\mu, \chi_{ij}^\dagger] = 0$ whenever $x = \pm ir$, where $L_a = \Sigma_i L_a(i)$ and $K_\mu = \Sigma_i K_\mu(i)$ are global generators. Thus, if each site is in the L representation, there are $2L$ quanta per site with $p=2L$ in Eq. (7). There are thus $4L$ quanta on each link. In the general SUSY AKLT state of Eq. (4), $2M$ of these quanta are passivated in singlet bonds. Thus, the maximum value of L_{ij} for the link is $J_{\max} = 2L - M$, where $L = \frac{1}{2}zM$ relates the value of L, the lattice coordination number z, and the integer parameter M. For the $L=1$ SVBS chain, for example, the wave function is annihilated by projectors onto either of the $L_{i,i+1} = \frac{3}{2}$ or $L_{i,i+1} = 2$ sectors and the only remaining possibilities are $L_{i,i+1} = 0$, $\frac{1}{2}$, and 1.

However, an inconvenient problem remains. Because the generators K_μ are non-Hermitian, this is also the case for the projection operators, and, thus, the Hamiltonian as well. Then, we use a "trick" to make a Hermitian Hamiltonian from non-Hermitian projection operators as performed in Sec. V. In Sec. V B, we also exhibit a properly Hermitian Hamiltonian which has the $L=1$ SVBS chain *with* fixed total fermion number an exact nondegenerate ground state. Before

doing so, we will derive the properties of the SVBS chains themselves.

III. SVBS SPIN CHAINS

The general SUSY AKLT chain wave function is written as the pair product,

$$|\Psi\rangle = \prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger + r f_i^\dagger f_{i+1}^\dagger)^M |\mathsf{V}\rangle. \tag{16}$$

This describes a chain in which each site is in the $L=M$ representation of $\mathsf{OSp}(1|2)$. The wave function is annihilated by projectors $\mathrm{P}_J(i,i+1)$ which project onto the total link angular momentum J, for $M<J\leq 2M$.

We are interested in computing correlation functions in these states. The correlation functions we will compute are

$$C_{\rm spin}(n) = \langle L(j)\cdot L(j+n)\rangle, \tag{17a}$$

$$C_{\rm SS}(n) = \langle (a_j b_{j+n} - b_j a_{j+n}) f_j^\dagger f_{j+n}^\dagger \rangle, \tag{17b}$$

corresponding to the spin-spin correlation function and a "singlet superconductivity" order-parameter correlation function. Since our state does not conserve particle number, the superconducting order parameter can be nonvanishing. Here $\langle\mathcal{O}\rangle = \langle\Psi|\mathcal{O}|\Psi\rangle / \langle\Psi|\Psi\rangle$ is the normalized expectation value. A corresponding "triplet superconductivity" correlator,

$$C_{\rm TS}^a(n) = \left\langle \begin{pmatrix} a_j a_{j+n} \\ \frac{1}{\sqrt{2}}(a_j b_{j+n} + b_j a_{j+n}) \\ b_j b_{j+n} \end{pmatrix} f_j^\dagger f_{j+n}^\dagger \right\rangle, \tag{18}$$

may also be defined. However, due to the singlet property of the SVBS states, we have that $C_{\rm TS}^a(l)=0$. We shall compute these correlations on finite chains, which have ends and examine the thermodynamic limit. There are some specific properties of edge states in these models, in direct correspondence to what is known from AKLT chains.[38–40] For example, the edges of the $L=1$ SVBS chain are local $L=\frac{1}{2}$ degrees of freedom, which means that the ground state of a long but finite $L=1$ SVBS chain will exhibit a ninefold quasidegeneracy, with the actual levels arranged into singlet, triplet, and quintuplet states, according to $\frac{1}{2}\otimes\frac{1}{2}=0\oplus\frac{1}{2}\oplus 1$.

Note that the operators whose correlation functions are computed must commute with the local constraint $n_a+n_b+n_f=p$. Expressions such as $\langle f_j^\dagger f_{j+n}^\dagger\rangle$ and $\langle a_j b_{j+n}\rangle$ vanish identically.

A. Spin-hole coherent states

The application of spin-coherent states in elucidating the properties of the AKLT VBS states was discussed in Ref. 8. Here we utilize a generalization of the familiar $\mathsf{SU}(2)$ spin-coherent states, known as spin-hole coherent states.[37] Define the state

$$|\hat{n},\theta;p\rangle \equiv \frac{1}{\sqrt{p!}}(u a^\dagger + v b^\dagger - \theta f^\dagger)^p |\mathsf{V}\rangle$$
$$= |\hat{n}\rangle_p \otimes |0\rangle - \sqrt{p}\,\theta |\hat{n}\rangle_{p-1}\otimes|1\rangle. \tag{19}$$

Here, $|\hat{n}_p\rangle$ is an $\mathsf{SU}(2)$ spin-coherent state with $S=\frac{1}{2}p$ and θ is a Grassmann variable which anticommutes with f and $f^\dagger$. The resolution of the identity may be written as

$$\int \frac{d\hat{n}}{4\pi}\int d\bar{\theta}\int d\theta e^{(p+1)\theta\bar{\theta}} |\hat{n},\theta;p\rangle\langle\hat{n},\theta;p| = P_{L=p/2}, \tag{20}$$

where P_L is the projector onto the angular momentum L representation of $\mathsf{OSp}(1|2)$.

Next, consider a general state in the angular momentum L representation written as

$$|\psi\rangle = \frac{1}{\sqrt{p!}}\psi(a^\dagger,b^\dagger,f^\dagger)|\mathsf{V}\rangle$$
$$= \frac{1}{\sqrt{p!}}[\psi_p(a^\dagger,b^\dagger) + \psi_{p-1}(a^\dagger,b^\dagger)f^\dagger]|\mathsf{V}\rangle, \tag{21}$$

where $\psi_p(a^\dagger,b^\dagger)$ is homogeneous of degree p in $a^\dagger$ and $b^\dagger$. We then have

$$\langle\hat{n},\theta;p|\psi\rangle = \psi(\bar{u},\bar{v},\bar{\theta}). \tag{22}$$

That is, we simply replace $a^\dagger\to\bar{u}$, $b^\dagger\to\bar{v}$, and $f^\dagger\to\bar{\theta}$ in the function ψ.

B. Matrix elements

Now consider the following spin operators:

$$\hat{T}_k^0 = \sum_{m=0}^{k}\sum_{n=0}^{k} T_{kmn}^0 a^m b^{k-m} a^{\dagger n} b^{\dagger k-n}, \tag{23a}$$

$$\hat{T}_k^+ = \sum_{m=0}^{k}\sum_{n=0}^{k+1} T_{kmn}^+ a^m b^{k-m} a^{\dagger n} b^{\dagger k+1-n}, \tag{23b}$$

$$\hat{T}_k^- = \sum_{m=0}^{k+1}\sum_{n=0}^{k} T_{kmn}^- a^m b^{k+1-m} a^{\dagger n} b^{\dagger k-n}. \tag{23c}$$

Note that $\hat{T}_k^\pm$ raise (+) or lower (−) the angular momentum by $\Delta L=\frac{1}{2}$, while $\hat{T}_k^0$ preserves total spin. Our goal is to compute the matrix element

$$\langle\psi|\hat{\mathcal{T}}_k|\phi\rangle = \frac{1}{p!}\langle\mathsf{V}|\bar{\psi}(a,b,f)[\hat{T}_k^0 + \hat{T}_k^+ f + \hat{T}_k^- f^\dagger$$
$$+ \hat{T}_k^{0\prime} f f^\dagger]\phi(a^\dagger,b^\dagger,f^\dagger)|\mathsf{V}\rangle \tag{24}$$

and to represent it as an integral over spin-hole coherent states. We find

$$\langle\psi|\hat{\mathcal{T}}_k|\phi\rangle = \int\frac{d\hat{n}}{4\pi}\int d\bar{\theta}\int d\theta e^{(p+1)\theta\bar{\theta}}$$
$$\times\,\bar{\psi}(u,v,\theta)\mathcal{T}_k(\hat{n},\theta,\bar{\theta})\phi(\bar{u},\bar{v},\bar{\theta}), \tag{25}$$

where

$$\mathcal{T}_k(\hat{n},\theta,\bar{\theta}) = \frac{(p+k+1)!}{p!}\left[\frac{T_k^0(\hat{n})}{p+k+1} + T_k^+(\hat{n})\theta + T_k^-(\hat{n})\bar{\theta} + T_k^{0'}(\hat{n})\theta\bar{\theta}\right]e^{k\theta\bar{\theta}} \tag{26}$$

replaces

$$\hat{\mathcal{T}}_k = \hat{T}_k^0 + \hat{T}_k^+ f + \hat{T}_k^- f^\dagger + \hat{T}_k^{0'} f f^\dagger. \tag{27}$$

C. Correlation functions

With the spin-hole coherent state formalism developed, we are now in position to calculate the correlation functions in the general SVBS chain state. The first step is to compute the wave-function normalization, which we call $\mathcal{D}$ (for "denominator"). Using the resolution of unity for the spin-hole coherent states, we have

$$\mathcal{D} = \langle\Psi|\Psi\rangle = \int d\mu \prod_{n=0}^{N} |u_n v_{n+1} - v_n u_{n+1} + r\theta_{n+1}\theta_n|^2, \tag{28}$$

where the measure is

$$d\mu = \prod_{j=0}^{N+1}\left[\frac{d\hat{n}_j}{4\pi} d\bar{\theta}_j d\theta_j\right] e^{(M+1)(\theta_0\bar{\theta}_0 + \theta_{N+1}\bar{\theta}_{N+1})} \times e^{(2M+1)(\theta_1\bar{\theta}_1 + \cdots \theta_N\bar{\theta}_N)}. \tag{29}$$

Note that the site $j=0$ and $j=N+1$, which are at the ends of the chain and have only one neighbor, are in the $L=\frac{1}{2}M$ representation of $\mathsf{OSp}(1|2)$ while the bulk sites are in the $L=M$ representation. We now expand

$$\begin{aligned}|u_n v_{n+1} - v_n u_{n+1} + r\theta_{n+1}\theta_n|^2 &= \left(\frac{1-\hat{n}_n\cdot\hat{n}_{n+1}}{2}\right)^M \\ &+ Mr\left(\frac{1-\hat{n}_n\cdot\hat{n}_{n+1}}{2}\right)^{M-1}(\bar{u}_n\bar{v}_{n+1} - \bar{v}_n\bar{u}_{n+1})\theta_{n+1}\theta_n \\ &+ M\bar{r}\left(\frac{1-\hat{n}_n\cdot\hat{n}_{n+1}}{2}\right)^{M-1}(u_n v_{n+1} - v_n u_{n+1})\bar{\theta}_n\bar{\theta}_{n+1} \\ &+ M^2|r|^2\left(\frac{1-\hat{n}_n\cdot\hat{n}_{n+1}}{2}\right)^{M-1}\theta_n\bar{\theta}_n\theta_{n+1}\bar{\theta}_{n+1}.\end{aligned} \tag{30}$$

Using

$$\int \frac{d\hat{n}}{4\pi}\left(\frac{1-\hat{n}\cdot\hat{n}'}{2}\right)^M = \frac{1}{M+1}, \tag{31}$$

we can now integrate out site $j=0$. The new integrand is then the truncated product wave function, starting with site $j=1$, multiplied by the quantity $\alpha_1+\beta_1\theta_1\bar{\theta}_1$, where $\alpha_1=1$ and $\beta_1 = M|r|^2$. The form of this expression self-replicates. That is, after integrating out sites $j=0$ through $j=n-1$ in succession, we are left with the expression $\alpha_n+\beta_n\theta_n\bar{\theta}_n$. We can now integrate out site n to obtain the replication formula,

$$\begin{aligned}\alpha_{n+1} + \beta_{n+1}\theta_{n+1}\bar{\theta}_{n+1} &\equiv \int\frac{d\hat{n}_n}{4\pi}\int d\bar{\theta}_n\int d\theta_n e^{(2M+1)\theta_n\bar{\theta}_n}(\alpha_n+\beta_n\theta_n\bar{\theta}_n) \\ &\times\left[\left(\frac{1-\hat{n}_n\cdot\hat{n}_{n+1}}{2}\right)^M + M^2|r|^2\left(\frac{1-\hat{n}_n\cdot\hat{n}_{n+1}}{2}\right)^{M-1} \times\theta_n\bar{\theta}_n\theta_{n+1}\bar{\theta}_{n+1}\right] \\ &= \left(\frac{2M+1}{M+1}\alpha_n + \frac{1}{M+1}\beta_n\right) + M|r|^2\alpha_n\theta_{n+1}\bar{\theta}_{n+1}.\end{aligned} \tag{32}$$

Note that in propagating the expression $(\alpha_n+\beta_n\theta_n\bar{\theta}_n)$, we may drop the last two terms on the right-hand side (RHS) of Eq. (30). We now have

$$\begin{pmatrix}\alpha_{n+1}\\ \beta_{n+1}\end{pmatrix} = \overbrace{\begin{pmatrix}\frac{2M+1}{M+1} & \frac{1}{M+1}\\ M|r|^2 & 0\end{pmatrix}}^{\mathcal{D}}\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}. \tag{33}$$

When we get to the last site, we have the final result

$$\mathcal{D} = \int\frac{d\hat{n}_{N+1}}{4\pi}\int d\bar{\theta}_{N+1}\int d\theta_{N+1}e^{(M+1)\theta_{N+1}\bar{\theta}_{N+1}} \times(\alpha_{N+1}+\beta_{N+1}\theta_{N+1}\bar{\theta}_{N+1}) = (M+1)\alpha_{N+1} + \beta_{N+1}. \tag{34}$$

Thus,

$$\mathcal{D} = (M+1\ \ 1)\mathcal{D}^N\begin{pmatrix}1\\ M|r|^2\end{pmatrix}. \tag{35}$$

Now we need to compute the numerator for the correlation function of interest.

1. Singlet superconductivity correlations

We define the singlet off-diagonal correlation function

$$C_{\mathrm{SS}}(n) = \left\langle\frac{1}{\sqrt{2}}(a_k b_{k+n} - b_k a_{k+n})f_k^\dagger f_{k+n}^\dagger\right\rangle, \tag{36}$$

which is independent of k in the limit of a long chain ($N\to\infty$). The operator above, in the language of the operators $\hat{T}_k^\sigma$ studied earlier, is of the form $\hat{T}_{k=0}^-$ on sites k and $k+n$. Invoking Eq. (26), we have

$$(a_k b_{k+n} - b_k a_{k+n})f_k^\dagger f_{k+n}^\dagger \to (2M+1)^2(u_k v_{k+n} - v_k u_{k+n})\bar{\theta}_k\bar{\theta}_{k+n}. \tag{37}$$

The correlation function may then be written

$$C_{\mathrm{SS}}(n) = \frac{1}{\sqrt{2}}(2M+1)^2\cdot\frac{\mathcal{N}}{\mathcal{D}}. \tag{38}$$

The calculation of the numerator $\mathcal{N}$ proceeds along the same lines as that of $\mathcal{D}$. Starting with site 0, we generate the expression $\alpha_j+\beta_n\theta_j\bar{\theta}_j$. When we arrive at site k, only the second term on the RHS of Eq. (30) contributes. We then have

$$\begin{aligned}
&\alpha_k\int\frac{d\hat{n}_k}{4\pi}\int d\bar{\theta}_k\int d\theta_k e^{(2M+1)\theta_k\bar{\theta}_k}\times(u_kv_{k+n}-v_ku_{k+n})\bar{\theta}_k\bar{\theta}_{k+n}\\
&\quad\times|u_kv_{k+1}-v_ku_{k+1}+r\theta_k\theta_{k+1}|^{2M}\\
&\quad=-\alpha_kMr\int\frac{d\hat{n}_k}{4\pi}\left(\frac{1-\hat{n}_k\cdot\hat{n}_{k+1}}{2}\right)^{M-1}\\
&\qquad\times(\bar{u}_k\bar{v}_{k+1}-\bar{v}_k\bar{u}_{k+1})\times(u_kv_{k+n}-v_ku_{k+n})\theta_{k+1}\bar{\theta}_{k+n}\\
&\quad=-\left(\frac{Mr}{M+1}\right)\alpha_k(\bar{u}_{k+1}u_{k+n}+\bar{v}_{k+1}v_{k+n})\theta_{k+1}\bar{\theta}_{k+n}. \qquad (39)
\end{aligned}$$

When we integrate over site $k+1$, we obtain

$$\left(\frac{M|r|}{M+1}\right)^2\alpha_k(u_{k+2}v_{k+n}-v_{k+2}u_{k+n})\bar{\theta}_{k+2}\bar{\theta}_{k+n}. \qquad (40)$$

We have now replicated the form of the integrand. Clearly whenever n is even, the numerator $\mathcal{N}$ vanishes. For n odd, we obtain

$$\beta_{k+n}=\left(\frac{M|r|}{M+1}\right)^n\delta_{n,\text{odd}}\cdot\alpha_k. \qquad (41)$$

The correlation length $\zeta(M,r)$ is then given by

$$\begin{aligned}
e^{-1/\zeta(M,r)}&=\frac{1}{\lambda_+}\left(\frac{M|r|}{M+1}\right)\\
&=\frac{M|r|}{M+\frac{1}{2}+\sqrt{\left(M+\frac{1}{2}\right)^2+M(M+1)|r|^2}}, \qquad (42)
\end{aligned}$$

where

$$\lambda_+=\frac{M+\frac{1}{2}}{M+1}+\sqrt{\left(\frac{M+\frac{1}{2}}{M+1}\right)^2+\frac{M|r|^2}{M+1}} \qquad (43)$$

is the largest eigenvalue of the matrix $\mathcal{D}$ from Eq. (33). We can define the s-wave order parameter as

$$\Delta=\langle(a_kb_{k+1}-b_ka_{k+1})f_k^\dagger f_{k+1}^\dagger\rangle=\frac{2M\left(M+\frac{1}{2}\right)^2r}{\left(\sqrt{M(M+1)(1+|r|^2)+\frac{1}{4}}+\frac{1}{2}\left(M+\frac{1}{2}\right)\right)^2-\frac{1}{4}\left(M+\frac{1}{2}\right)^2}. \qquad (44)$$

(Δ is plotted in Fig. 2.)

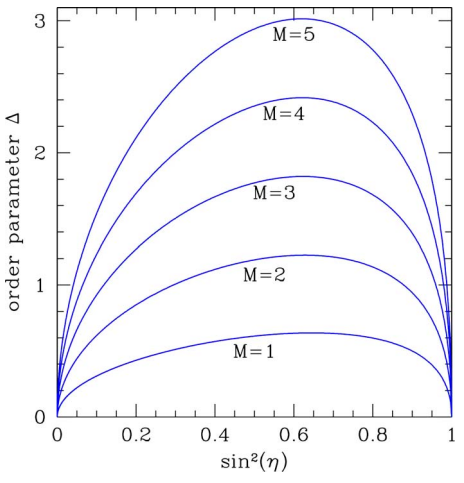

FIG. 2. (Color online) Order parameter $\Delta=\langle(a_nb_{n+1}-b_na_{n+1})f_n^\dagger f_{n+1}^\dagger\rangle$ in the general SVBS chain. The parameter η is given by $\eta=\tan^{-1}|r|$.

2. Spin correlations

We next turn to the spin-spin correlation function, $C_{\text{spin}}(n)=\langle L(j)\cdot L(j+n)\rangle$. The spin operator is given by $L=\frac{1}{2}b_\mu^\dagger\sigma_{\mu\nu}b_\nu$ and is of the type $\hat{T}^0_{k=1}$. Accordingly, Eq. (26) gives the prescription

$$L(j)\to\left(M+\frac{1}{2}\right)\hat{n}_je^{\theta_j\bar{\theta}_j}. \qquad (45)$$

Once again, the correlation function is expressed as a ratio of $\mathcal{N}/\mathcal{D}$. In the numerator, when we arrive at site k, we have the integral

$$\begin{aligned}
&\left(M+\frac{1}{2}\right)\int\frac{d\hat{n}_k}{4\pi}\int d\bar{\theta}_k\int d\theta_k e^{2(M+1)\theta_k\bar{\theta}_k}\\
&\quad\times(\alpha_k+\beta_k\theta_k\bar{\theta}_k)\hat{n}_k\left[\left(\frac{1-\hat{n}_k\cdot\hat{n}_{k+1}}{2}\right)^M\right.\\
&\quad\left.+M^2|r|^2\left(\frac{1-\hat{n}_k\cdot\hat{n}_{k+1}}{2}\right)^{M-1}\theta_k\bar{\theta}_k\theta_{k+1}\bar{\theta}_{k+1}\right]
\end{aligned}$$

$$= (\alpha_{k+1} + \beta_{k+1}\theta_{k+1}\bar{\theta}_{k+1})\hat{n}_{k+1}, \tag{46}$$

with

$$\begin{pmatrix} \alpha_{k+1} \\ \beta_{k+1} \end{pmatrix} = -\frac{M(M+\frac{1}{2})}{M+1} \begin{pmatrix} \frac{2(M+1)}{M+2} & \frac{1}{M+2} \\ (M-1)|r|^2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}. \tag{47}$$

For sites j between k and $k+n$, we have

$$\int \frac{d\hat{n}_j}{4\pi} \int d\bar{\theta}_j \int d\theta_j e^{(2M+1)\theta_j\bar{\theta}_j} \times (\alpha_j + \beta_j\theta_j\bar{\theta}_j)\hat{n}_j \left[\left(\frac{1-\hat{n}_j\cdot\hat{n}_{j+1}}{2} \right)^M + M^2|r|^2 \left(\frac{1-\hat{n}_j\cdot\hat{n}_{j+1}}{2} \right)^{M-1} \theta_j\bar{\theta}_j\theta_{j+1}\bar{\theta}_{j+1} \right] = (\alpha_{j+1} + \beta_{j+1}\theta_{j+1}\bar{\theta}_{j+1})\hat{n}_{j+1}, \tag{48}$$

with

$$\begin{pmatrix} \alpha_{j+1} \\ \beta_{j+1} \end{pmatrix} = \mathcal{K} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}, \tag{49}$$

where

$$\mathcal{K} = -\frac{M}{(M+1)(M+2)} \begin{pmatrix} 2M+1 & 1 \\ (M-1)(M+2)|r|^2 & 0 \end{pmatrix}. \tag{50}$$

Finally, we come to site $k+n$, where we have $\hat{n}_{k+n}\cdot\hat{n}_{k+n}=1$ and

$$\left(M+\frac{1}{2}\right) \int \frac{d\hat{n}_{k+n}}{4\pi} \int d\bar{\theta}_{k+n} \int d\theta_{k+n} e^{2(M+1)\theta_{k+n}\bar{\theta}_{k+n}} \times (\alpha_{k+n} + \beta_{k+n}\theta_{k+n}\bar{\theta}_{k+n}) \left[\left(\frac{1-\hat{n}_{k+n}\cdot\hat{n}_{k+n+1}}{2} \right)^M + M^2|r|^2 \left(\frac{1-\hat{n}_{k+n}\cdot\hat{n}_{k+n+1}}{2} \right)^{M-1} \times \theta_{k+n}\bar{\theta}_{k+n}\theta_{k+n+1}\bar{\theta}_{k+n+1} \right] = (\alpha_{k+n+1} + \beta_{k+n+1}\theta_{k+n+1}\bar{\theta}_{k+n+1}), \tag{51}$$

with

$$\begin{pmatrix} \alpha_{k+n+1} \\ \beta_{k+n+1} \end{pmatrix} = \left(M+\frac{1}{2}\right) \begin{pmatrix} 2 & \frac{1}{M+1} \\ M|r|^2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{k+n} \\ \beta_{k+n} \end{pmatrix}. \tag{52}$$

For sites $l>k+n$, we propagate by the matrix $\mathcal{D}$ from the denominator.

Assuming $N\to\infty$, with n finite but large, we can ignore the ends and we obtain

$$C_{\text{spin}}(n) = A(\lambda_{\mathcal{K}}/\lambda_{\mathcal{D}})^{|n|} = A(-1)^n e^{-|n|/\xi(M,r)}, \tag{53}$$

where A is a coefficient and $\lambda_{\mathcal{K},\mathcal{D}}$ are the largest magnitude eigenvalues of the matrices $\mathcal{K}$ and $\mathcal{D}$, respectively. The spin-correlation length is thus given by

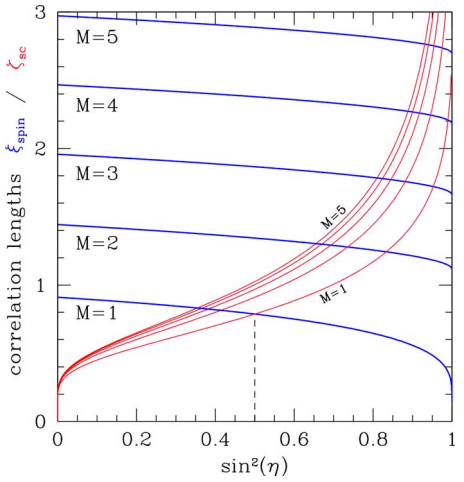

FIG. 3. (Color online) Spin correlation length $\xi(M,r)$ (blue) and superconducting correlation length $\zeta(M,r)$ (red) for the general SUSY AKLT chain.

$$e^{-1/\xi(M,r)} = -\frac{\lambda_{\mathcal{K}}}{\lambda_{\mathcal{D}}} = \frac{M}{M+2} \cdot \left(\frac{1+\sqrt{1+\frac{(M-1)(M+2)|r|^2}{\left(M+\frac{1}{2}\right)^2}}}{1+\sqrt{1+\frac{M(M+1)|r|^2}{\left(M+\frac{1}{2}\right)^2}}} \right). \tag{54}$$

In the $r\to 0$ limit we recover the result $C(n)=A(-\frac{M}{M+2})^{|n|}$ found for general AKLT chains in Ref. 41. Note that for $r\to\infty$ and $M=1$ the correlation length vanishes. This is because in this limit the ground state is that for the $S=\frac{1}{2}$ Majumdar-Ghosh model, i.e., alternating singlets, for which there are no correlations beyond nearest neighbors. For the $M>1$ generalizations of Majumdar-Ghosh, however, the correlation length is finite. The spin-correlation length $\xi(M,r)$ and superconducting correlation length $\zeta(M,r)$ are both plotted in Fig. 3 versus the parameter $\sin^2\eta \equiv |r|^2/(1+|r|^2)$. These two correlation lengths coincide at $r=(2M+1)/3$. Especially, when $M=1$, they coincide at $r=1$.

IV. RELATION TO QHE STATES

Here, we discuss analogies between the lowest Landau-level (LLL) physics and the spin physics in detail. Much of our discussion is an extension of the pioneering work by Haldane[10] on the FQHE in a spherical geometry.

We begin with a discussion about analogies in one-particle problem. For The LLL bases are given by the monopole harmonics,[42] which form an irreducible representation of SU(2) indexed by the unique Casimir operator, which is the monopole charge. As is well known, the monopole harmonics in the LLL are constructed as symmetric products of Hopf spinors. Mathematically, the Hopf spinor is equivalent

TABLE I. Correspondences between LLL physics and spin physics.

	LLL physics	Spin physics
Space	External	Internal
Quantum number	Monopole charge	Spin magnitude
Basic state	Hopf spinor	Spin-coherent state
Manifold	Fuzzy sphere	Bloch sphere

to a spin-coherent state for a state in the fundamental ($S=\frac{1}{2}$) representation. The symmetric products of the spin-coherent states give rise to higher spin states. In the LLL, the kinetic term is quenched and the coordinates of the two spheres are effectively reduced to operators of SU(2) algebra. Such manifold with noncommutative coordinates is known as the fuzzy sphere and its mathematical structure is equal to the Bloch sphere of spin physics. The relations between the LLL states and the spin states are summarized in Table I. Thus, as for the one-particle problem, there are apparent analogies between the LLL physics and the spin physics.

A. Laughlin-Haldane and AKLT states

Even in many-body level, as briefly mentioned in Sec. I, remarkable resemblances between the Laughlin state and the AKLT state have been reported in the work of one of the authors.[8] On Haldane's two spheres, particles are uniformly distributed to form a rotationally invariant incompressible liquid described by the Laughlin-Haldane function,

$$\Phi_{\mathrm{LH}}^{(m)}=\prod_{i<j}^{N}(u_iv_j-v_iu_j)^m, \tag{55}$$

where (u,v) indicates the Hopf spinor. Meanwhile, the AKLT state is the VBS state made by the SU(2) singlet combination of Schwinger bosons [Eq. (1)], and, in the spin-coherent state representation, is written as

$$\Phi_{\mathrm{AKLT}}^{(M)}=\prod_{\langle ij\rangle}^{z}(u_iv_j-v_iu_j)^M. \tag{56}$$

Obvious resemblances may be found between Eqs. (55) and (56). The power m in the Laughlin-Haldane state takes even or odd integer depending on the statistics of the particles, while M in the AKLT state specifies the number of the valence bonds on each site and has nothing to do with statistics. Since these two states are "almost" mathematically equivalent, their truncated pseudopotential Hamiltonians are similarly constructed by the form of two-body interactions: the truncated Hamiltonian for the AKLT state is given by Eq. (3), while for the Laughlin-Haldane state, it is given by

$$H=\sum_{i<j}\sum_{J^*+1}^{2S}V_JP_J(i,j), \tag{57}$$

where $J^*=2S-m$ with $S=m(N-1)/2$. Based on the OSp(1|2) supergroup analysis, the SUSY Laughlin-Haldane wave function was proposed as

$$\Psi_{\mathrm{SLH}}^{(m)}=\prod_{i<j}^{N}(u_iv_j-v_iu_j+r\theta_i\theta_j)^m, \tag{58}$$

where (u,v,θ) indicates the SUSY Hopf spinor. In Ref. 26, r is fixed as -1 but here we take r as a free parameter. Extracting the original Laughlin-Haldane wave function, the SUSY Laughlin-Haldane state can be rewritten as

$$\Psi_{\mathrm{SLH}}^{(m)}=\exp\left(mr\sum_{i<j}\frac{\theta_i\theta_j}{u_iv_j-v_iu_j}\right)\cdot\Phi_{\mathrm{LH}}^{(m)}. \tag{59}$$

All of the important physics are included in the exponential factor of Eq. (59) and this deformation enables us to perform an intuitive interpretation of the SUSY Laughlin-Haldane wave function. The denominator of the exponential factor, $1/(u_iv_j-v_iu_j)$, represents a p-wave bound state of two particles i and j, and the SUSY Laughlin state is regarded as a p-wave superfluid on the original Laughlin state.[28] By expanding the exponent, one may find

$$\begin{aligned}\Psi_{\mathrm{SLH}}^{(m)}=&\Phi_{\mathrm{LH}}^{(m)}+mr\left(\sum_{i<j}^{N}\frac{\theta_i\theta_j}{u_iv_j-v_iu_j}\right)\cdot\Phi_{\mathrm{LH}}^{(m)}\\&+\frac{1}{2}(mr)^2\left(\sum_{i<j}^{N}\frac{\theta_i\theta_j}{u_iv_j-v_iu_j}\right)^2\cdot\Phi_{\mathrm{LH}}^{(m)}+\cdots\\&+(mr)^{N/2}\prod_{i}^{N}\theta_i\cdot\mathcal{A}\left(\prod_{j:\mathrm{even}}\frac{1}{u_{j-1}v_j-v_{j-1}u_j}\right)\cdot\Phi_{\mathrm{LH}}^{(m)},\end{aligned} \tag{60}$$

where $\mathcal{A}$ in the last term represents antisymmetrization over all different choices of breaking particles into pairs and is simply known as the Pfaffian. Hence, the last term in Eq. (60) represents the Pfaffian state proposed by Moore and Read[43]

$$\Phi_{\mathrm{MR}}^{(m)}=\mathcal{A}\prod_{i:\mathrm{even}}\frac{1}{u_{i-1}v_i-v_{i-1}u_i}\cdot\Phi_{\mathrm{LH}}^{(m)}=\mathrm{Pf}\left(\frac{1}{u_iv_j-v_iu_j}\right)\cdot\Phi_{\mathrm{LH}}^{(m)}, \tag{61}$$

where all of the particles form p-wave pairings to form a bosonic QH state. It is noted that the expression (60) should be regarded as the expansion about the parameter r not m since the original Laughlin-Haldane function itself depends on m.

B. Physical interpretation of the SVBS state

Inspired by the similarity between the Laughlin-Haldane and the AKLT states, from the SUSY Laughlin-Haldane wave function [Eq. (58)], one may derive the SUSY AKLT state,

$$\Psi_{\mathrm{AKLT}}^{(M)}=\prod_{\langle ij\rangle}^{z}(u_iv_j-v_iu_j+r\eta_i\eta_j)^M, \tag{62}$$

which is the spin-hole coherent representation of Eq. (4). In the following, we focus on the SVBS spin chain. Just as in the SUSY Laughlin-Haldane case, the SVBS spin chain state $z=2$ is rewritten as

FIG. 4. (Color online) The single-bond breaking operator annihilates a valence bond and creates a fermion pair on the nearest-neighbor sites.

$$\Psi_{\mathrm{AKLT}}^{(M)} = \exp\left(Mr \sum_i \frac{\theta_i \theta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}} \right) \cdot \Phi_{\mathrm{AKLT}}^{(M)}, \quad (63)$$

where the exponential factor $\theta_i\theta_j/(u_iv_{i+1}-v_iu_{i+1})$ which we call the "pair creator" has the following physical interpretation: it replaces one of the valence bonds between sites i and $i+1$ by a fermion (hole) pair; this is depicted in Fig. 4.

The SVBS chain state is expanded as

$$\begin{aligned}\Psi_{\mathrm{AKLT}}^{(M)} &= \Phi_{\mathrm{AKLT}}^{(M)} + Mr\left(\sum_i \frac{\theta_i \theta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}} \right) \cdot \Phi_{\mathrm{AKLT}}^{(M)} \\ &+ \frac{1}{2}(Mr)^2 \left(\sum_i \frac{\theta_i \theta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}} \right)^2 \cdot \Phi_{\mathrm{AKLT}}^{(M)} + \cdots \\ &+ (Mr)^{L/2} \prod_j \theta_j \left(\prod_{i \atop \text{even}} - \prod_{i \atop \text{odd}} \right) \frac{1}{u_i v_{i+1} - v_i u_{i+1}} \cdot \Phi_{\mathrm{AKLT}}^{(M)}. \end{aligned} \quad (64)$$

We assume here that the total number of sites L in our ring is even. The original AKLT state appears as the first term in this expansion in powers of the Grassmann coordinates. The second term consists of superpositions of all AKLT states with one hole pair, the third term of all superpositions with two hole pairs, etc. The final term in the expansion contains the product $\theta_1 \cdots \theta_L$ over all sites. Its corresponding spin wave function is a superposition of two generalized Majumdar-Ghosh states, one in which a valence bond has been removed from each even link $(2n, 2n+1)$ and the other where a valence bond has been removed from each odd link $(2n-1, 2n)$. Note that each site can accommodate at most one hole (Fig. 5).

As discussed in Sec. I, for $M=1$ the last term of Eq. (64) gives precisely the $S=\frac{1}{2}$ Majumdar-Ghosh state,

$$\begin{aligned}&\left(\prod_{i \atop \text{even}} - \prod_{i \atop \text{odd}} \right) \frac{1}{u_i v_{i+1} - v_i u_{i+1}} \cdot \Phi_{\mathrm{AKLT}}^{(M=1)} \\ &= \left(\prod_{i \atop \text{even}} - \prod_{i \atop \text{odd}} \right) (u_i v_{i+1} - v_i u_{i+1}) = \Phi_A - \Phi_B, \end{aligned} \quad (65)$$

where Φ_A and Φ_B correspond to the two dimer states of Eq. (5).

Thus, both the Majumdar-Ghosh and Moore-Read states appear as the last terms in the expansion of the corresponding super wave functions. It is interesting to note in this regard that both the Majumdar-Ghosh and Moore-Read wave functions vanish when three particles (Moore-Read) or any three neighboring spins (Majumdar-Ghosh) coincide and

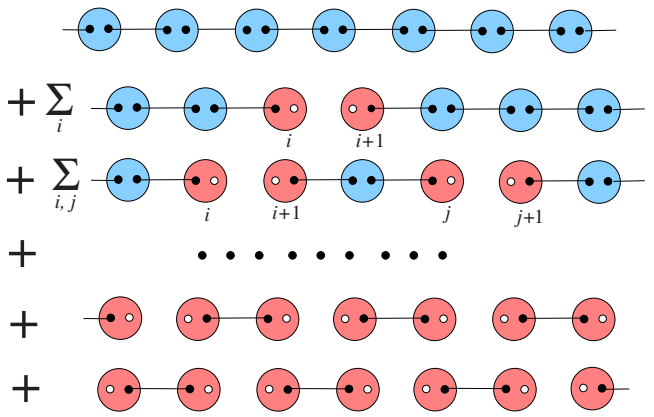

FIG. 5. (Color online) Graphical representation for the expansion of the SUSY AKLT spin state with $M=1$ [Eq. (64)]. At the n^{th} term of the expansion, there appears the superposition of AKLT states with $(n-1)$ hole pairs. In particular, the original AKLT state is realized as the first term $(r \to 0)$ and the MG dimer states are realized as the last term $(r \to \infty)$.

their truncated pseudopotential Hamiltonians are constructed by three-body interactions.[44] For the Moore-Read state,

$$H_{\mathrm{MR}} = \sum_{i<j<k} \sum_{J=3(S-m)+2}^{3S} V_J P_J(i,j,k) \quad (66)$$

with $S=\frac{1}{2}[m(N-1)-1]$, while for the Majumdar-Ghosh state

$$H_{\mathrm{MG}} = V_{3/2} \sum_i P_{3/2}(i, i+1, i+2). \quad (67)$$

C. More fermion coordinates

Our construction may be generalized to include additional Grassmann coordinates. Introducing two Grassmann species θ_i and η_i, we write the extended SUSY Laughlin-Haldane wave function as

$$\Psi_{\mathrm{SLH}}^{(m)} = \prod_{i<j} (u_i v_j - v_i u_j + r_1 \theta_i \theta_j + r_2 \eta_i \eta_j)^m, \quad (68)$$

where r_1 and r_2 are two free parameters. We may now write

$$\begin{aligned}\Psi_{\mathrm{SLH}}^{(m)} &= \exp\left(mr_1 \sum_{i<j} \frac{\theta_i \theta_j}{u_i v_j - v_i u_j} \right) \cdot \exp\left(mr_2 \sum_{i<j} \frac{\eta_i \eta_j}{u_i v_j - v_i u_j} \right) \\ &\cdot \exp\left(- mr_1 r_2 \sum_{i<j} \frac{\theta_i \theta_j \eta_i \eta_j}{(u_i v_j - v_i u_j)^2} \right) \cdot \Phi_{\mathrm{SLH}}^{(m)}. \end{aligned} \quad (69)$$

We have already encountered the first and second exponents of Eq. (69) in the previous analysis, each of which represents the p-wave pairing state. The third exponent is the newly appeared term and its exponential factor provides $(-1)^2$ by the interchange of i and j to suggest the property of d-wave pairing. When we expand the third exponent, at the last term, we obtain

$$\int \prod_i^N d\theta_i d\eta_i \cdot \exp\left(-mr_1 r_2 \sum_{i<j} \frac{\theta_i\theta_j\eta_i\eta_j}{(u_i v_j - v_i u_j)^2}\right) = (-mr_1 r_2)^{N/2} S\left(\frac{1}{(u_i v_j - v_i u_j)^2}\right), \quad (70)$$

where $\Pi_i d\theta_i d\eta_i \equiv \Pi_i d\theta_i \Pi_i d\eta_i$ and S represents the symmetrization operation, which is realized by changing all the signs of terms in Pfaffian to be plus and is known to yield the Haffnian,

$$S\left(\frac{1}{(u_i v_j - v_i u_j)^2}\right) = \mathrm{Hf}\left(\frac{1}{(u_i v_j - v_i u_j)^2}\right). \quad (71)$$

The first and second exponents in Eq. (69) are expanded as in Eq. (60) to yield the product of two Pfaffians and produce the Haffnian again,

$$(mr_1)^{N/2}(mr_2)^{N/2}\,\mathrm{Pf}^2\left(\frac{1}{(u_i v_j - v_i u_j)^2}\right) = m^N (r_1 r_2)^{N/2}\,\mathrm{Hf}\left(\frac{1}{(u_i v_j - v_i u_j)^2}\right). \quad (72)$$

Besides this, there are many cross terms to yield Haffnian in the products of expansions of the three exponents. Collecting all of the contributions, the last term of the expansion [Eq. (69)] is summarized as

$$\int \prod_i d\theta_i d\eta_i \Psi_{\mathrm{SLH}}^{(m)} = [m(m-1)r_1 r_2]^{N/2} \cdot \Phi_{\mathrm{HR}}^{(m)}, \quad (73)$$

where Φ_{HR} is the Haffnian state of Haldane-Rezayi,[45]

$$\Phi_{\mathrm{HR}}^{(m)} = \mathrm{Hf}\left[\frac{1}{(u_i v_j - v_i u_j)^2}\right] \cdot \Phi_{\mathrm{SLH}}^{(m)}, \quad (74)$$

which represents d-wave pairing QH state. We see that the Laughlin wave function with two Grassmann species [Eq. (69)] is expanded as

$$\Psi_{\mathrm{SLH}}^{(m)} = \Phi_{\mathrm{LH}}^{(m)} + \cdots + \left[(mr_1)^{N/2}\prod_i \theta_i + (mr_2)^{N/2}\prod_i \eta_i\right] \cdot \Phi_{\mathrm{MR}}^{(m)} + \cdots + [m(m-1)r_1 r_2]^{N/2}\prod_i \theta_i\eta_i \cdot \Phi_{\mathrm{HR}}^{(m)}. \quad (75)$$

Intriguingly, with two species of Grassmann coordinates, there appear Laughlin, Moore-Read, and Haldane-Rezayi states as expansion coefficients. Each of them naturally appears in the following limits: the Laughlin state at $r_1, r_2 \to 0$, the Moore-Read state at $r_1 \to \infty$ or $r_2 \to \infty$ with $r_1 r_2$ fixed, and the Haldane-Rezayi state at $r_1, r_2 \to \infty$. Now, let us move to the discussion of the VBS model with two species of Grassmann coordinates. The corresponding generalized AKLT state is

$$\Psi_{\mathrm{AKLT}}^{(M)} = \prod_{\langle ij\rangle}^{z} (u_i v_j - v_i u_j + r_1\theta_i\theta_j + r_2\eta_i\eta_j)^M, \quad (76)$$

and, for 1D spin chain, it is rewritten

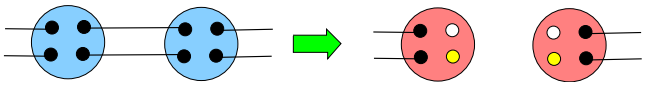

FIG. 6. (Color online) The operation of the double-bond breaking operator. The white circles represent the hole pair of $\theta_i\theta_{i+1}$, while the light yellow circles represent the other hole pair of $\eta_i\eta_{i+1}$.

$$\Psi_{\mathrm{AKLT}}^{(M)} = \Phi_{\mathrm{AKLT}}^{(M)} \cdot \exp\left(Mr_1 \sum_i \frac{\theta_i\theta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}}\right) \cdot \exp\left(Mr_2 \sum_i \frac{\eta_i\eta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}}\right) \cdot \exp\left[-Mr_1 r_2 \sum_i \frac{\theta_i\theta_{i+1}\eta_i\eta_{i+1}}{(u_i v_{i+1} - v_i u_{i+1})^2}\right]. \quad (77)$$

In the following, we concentrate on the case $M=2$. The factor of the third exponent $\theta_i\theta_{i+1}\eta_i\eta_{i+1}/(u_i v_{i+1} - v_i u_{i+1})^2$ is interpreted as the "double-bond breaking operator:" it annihilates two valence bonds and creates two kinds of fermion pairs between i and $i+1$ sites (Fig. 6).

Then, in Eq. (77), there are two types of bond breaking operations, one of which is the single-bond breaking operations performed by first and second exponents and the other is the double-bond breaking operation by the third exponent. With this interpretation, we have a nice graphical understanding of the expansion of the generalized AKLT state (see Fig. 7).

As expected from the graphical representation, in the last terms of the order of $(r_1 r_2)^{L/2}$ there appear two fully dimerized states and two partially dimerized states. An explicit calculation yields

$$\int \prod_i d\theta_i d\eta_i \Psi_{\mathrm{AKLT}}^{(M=2)} = (2r_1 r_2)^{L/2}\left(\prod_{i\ \mathrm{even}} + \prod_{i\ \mathrm{odd}}\right)(u_i v_{i+1} - v_i u_{i+1})^2 - 2^{L+1}(r_1 r_2)^{L/2}\prod_i (u_i v_{i+1} - v_i u_{i+1}), \quad (78)$$

where once again we consider a ring of L sites with L even. Equation (78) corresponds to the expression (73) of the QHE. The first two terms on the RHS in Eq. (78) denote the two fully dimerized states, while the last term on the RHS represents the two partially dimerized states. These fully and partially dimerized states are degenerate zero-energy eigenstates of the three-body truncated pseudopotential Hamiltonian,

$$H_D = \sum_i \sum_{J=2}^{3} V_J P_J(i, i+1, i+2). \quad (79)$$

The degeneracies may be resolved by adding terms involving other projection operators to the Hamiltonian.[7] Since the fully dimerized states in Eq. (78) only take the spin magnitude $J=1$ for groups of three consecutive sites, they are the zero-energy eigenstates of the Hamiltonian,

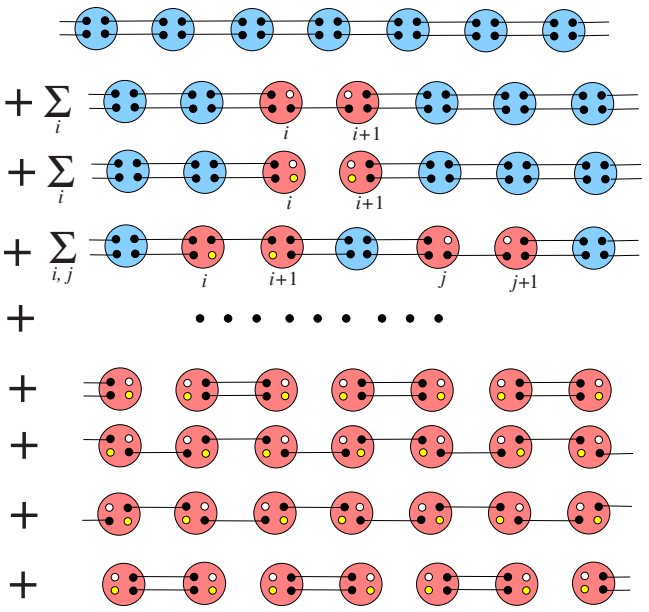

FIG. 7. (Color online) The graphical representation for the expansion of the generalized AKLT spin chain state [Eq. (77)]. The first term represents the original $M=2$ AKLT state. At the second term, the superposition of the AKLT states with one hole pair appears. At both third and fourth terms, one may find the AKLT states with two hole pairs. At the third term, the two holes are generated by the double-bond breaking operation, while at the fourth term, they are generated by two successive different single-bond breaking operations. At the last terms of the expansion, we obtain four states, two of which are fully dimerized states and the other two are partially dimerized states that are equal to the $M=1$ AKLT states.

$$H_{FD} = H_D + \sum_i V_0 P_0(i,i+1,i+2)$$

$$= \sum_i \sum_{J\neq 1} V_J P_J(i,i+1,i+2), \tag{80}$$

while the partially dimerized states are not.

Comparing the two expressions (78) and (73), one notices the apparent analogies between the fully dimerized double-bond states and the HR state. As in the case of the dimerized single-bond state and the MR state, they share common features such as the truncated pseudopotential Hamiltonians which render them exact ground states. For the fully dimerized state, the Hamiltonian is given by the three-body interaction form (80), while for the HR state, it has a similar form,

$$H_{\mathrm{HR}} = \sum_{i<j} \sum_{J=3(S-m)+3}^{3S} V_J P_J(i,j,k), \tag{81}$$

with $S=\frac{1}{2}[m(N-1)-2]$.

The generalization with more fermionic coordinates is a straightforward task. With F species of fermionic coordinates, the SUSY AKLT state is generalized as

$$\Psi^{(M)}_{\mathrm{AKLT}} = \prod_{\langle ij\rangle} \left(u_i v_j - v_i u_j + \sum_{f=1}^{F} r_f \theta_i^f \theta_j^f \right)^M, \tag{82}$$

and is rewritten as

$$\Psi^{(M)}_{\mathrm{AKLT}} = \exp\left(M \sum_f^F r_f \sum_{\langle ij\rangle} \frac{\theta_i^f \theta_j^f}{u_i v_j - v_i u_j} \right)$$
$$\times \exp\left[-M \sum_{f<f'}^{F} r_f r_f' \sum_{\langle ij\rangle} \frac{\theta_i^f \theta_j^f \theta_i^{f'} \theta_j^{f'}}{(u_i v_j - v_i u_j)^2} \right]$$
$$\times \exp\left[2M \sum_{f<f'<f''}^{F} r_f r_{f'} r_{f''} \sum_{\langle ij\rangle} \frac{\theta_i^f \theta_j^f \theta_i^{f'} \theta_j^{f'} \theta_i^{f''} \theta_j^{f''}}{(u_i v_j - v_i u_j)^3} \right]$$
$$\cdots \exp\left[\frac{(-1)^{F-1}(F-1)!\, M r_1 r_2 \cdots r_F}{\cdot \sum_{\langle ij\rangle} \frac{\theta_i^1 \theta_j^1 \theta_i^2 \theta_j^2 \cdots \theta_i^F \theta_j^F}{(u_i v_j - v_i u_j)^F}} \right] \cdot \Phi^{(M)}_{\mathrm{AKLT}}. \tag{83}$$

As in the previous discussion, we consider the expansion of the exponentials in Eq. (83). At the first term of the expansion, we obtain the original AKLT state with $S=\frac{1}{2}zM$. The last terms, of order $(M^F r_1 r_2 \cdots r_F)^{L/2}$, represent a nearest-neighbor RVB state with $S=\frac{1}{2}(zM-F)$. For the SVBS spin chain, the last terms are (fully and partially) dimerized states that are degenerate zero-energy eigenstates of the three-body interaction Hamiltonian,

$$H_D = \sum_i \sum_{J=S+1}^{3S} V_J P_J(i,i+1,i+2), \tag{84}$$

with $S=\frac{1}{2}(2M-F)$. When $M=F$, the two degenerate fully dimerized states appear in the last terms and are the zero-energy eigenstates of the truncated Hamiltonian,

$$H_{FD} = \sum_i \sum_{J\neq S}^{3S} V_J P_J(i,i+1,i+2), \tag{85}$$

with $S=\frac{1}{2}M$.

D. BCS aspects of the SVBS state

In Sec. IV, we have mainly discussed the property of the SVBS state in the two limits $r\to 0,\infty$ and found that the $M=1$ SVBS spin chain produces the original AKLT state at $r\to 0$, while the MG state at $r\to\infty$. With finite r, the SVBS state contains a finite density of hole pairs and accordingly exhibits superconducting properties. This state of affairs is familiar from the BCS state,

$$|\mathrm{BCS}\rangle = \prod_k \frac{1}{\sqrt{1+|g_k|^2}} (1 + g_k c_k^\dagger c_{-k}^\dagger)|0\rangle. \tag{86}$$

As $g_k\to 0$, the BCS state is reduced to the vacuum, while at $g_k\to\infty$, it becomes the completely filled Fermi sphere. For intermediate g_k, the $|\mathrm{BCS}\rangle$ describes a state with off-diagonal long-ranged order. Then, one may conjecture the following correspondences:

$$g_k \leftrightarrow r, \quad |0\rangle \leftrightarrow \Phi_{\mathrm{AKLT}}, \quad |F\rangle \leftrightarrow \Phi_{\mathrm{MG}}. \tag{87}$$

Interestingly, the BCS state exhibits a duality (S duality, in terminology of high-energy theory) with respect to the coherence factor,

$$g_k \leftrightarrow \frac{1}{g_k^*}. \tag{88}$$

To see this, it is important to notice that the BCS state is represented in two ways,

$$|\mathrm{BCS}\rangle = \prod_k \frac{1}{\sqrt{1+|g_k|^2}} \exp(g_k c_k^\dagger c_{-k}^\dagger)|0\rangle$$

$$= \prod_k \frac{1}{\sqrt{1+|g_k|^{-2}}} \exp(g_k^{-1} h_k^\dagger h_{-k}^\dagger)\ |0\rangle\rangle, \tag{89}$$

where h_k represents the hole operator $h_k^\dagger = c_{-k}$ and $|0\rangle\rangle$ is the hole vacuum with $h_k\ |0\rangle\rangle = 0$, namely, the fully occupied Fermi sphere $|0\rangle\rangle = |\mathrm{F}\rangle$. As seen in Eq. (89), the two descriptions in terms of particle and hole operators are completely equivalent and the duality physically represents the particle-hole symmetry. The order parameter

$$\Delta_k = \langle c_k^\dagger c_{-k}^\dagger \rangle = \frac{g_k^*}{1+|g_k|^2} = \frac{1}{g_k + g_k^{*-1}}, \tag{90}$$

manifestly reflects the dual structure of Eq. (88). The order parameter thus vanishes in two limits: the weak limit $g_k \to 0$ and the strong limit $g_k \to \infty$. It takes its maximum value at the self-dual point $|g_k| = 1$. The average occupancy of the momentum k state and its fluctuation are given by

$$\langle n_k \rangle = \frac{|g_k|^2}{1+|g_k|^2}, \tag{91a}$$

$$\langle (n_k - \langle n_k \rangle)^2 \rangle = \frac{|g_k|^2}{(1+|g_k|^2)^2} = \frac{1}{(g_k + g_k^{*-1})(g_k^* + g_k^{-1})}. \tag{91b}$$

The fluctuation, too, is maximalized at the self-dual point $|g_k| = 1$. As the duality is manifest in the BCS state and especially between $|0\rangle$ and $|\mathrm{F}\rangle$, one may speculate a hidden duality between the AKLT state and the MG state

$$\Phi_{\mathrm{AKLT}} \overset{\text{dual?}}{\leftrightarrow} \Phi_{\mathrm{MG}}. \tag{92}$$

Indeed, the parameter-dependent terms in $\mathsf{OSp}(1|2)$ Casimir operator, Eq. (13), are given by

$$-\frac{1}{4r}(a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger) f_i f_j - \frac{r}{4}(a_i b_j - b_i a_j) f_i^\dagger f_j^\dagger, \tag{93}$$

which implies a duality

$$r \leftrightarrow \frac{1}{r}, \quad a_i b_j - b_i a_j \leftrightarrow f_i f_j. \tag{94}$$

This is also the case *vis-a-vis* the truncated pseudopotential Hamiltonians for the SVBS states. Physically, this duality corresponds to the interchange of VB and fermion pair, in which case the SVBS state of Eq. (4) is obviously invariant under the dual transformation. Though the VB and the fermion pair operators possess same antisymmetric property with interchange of i and j, their squares exhibit different properties: the square of the VB is nonzero, while the fermion pair vanishes. More typically, we cannot naively take the limit $r \to \infty$ in the SVBS state since in that limit, the SVBS state becomes

$$\Psi_{\mathrm{AKLT}} \to \prod_{\langle ij \rangle} \eta_i \eta_j = 0, \tag{95}$$

unlike the BCS state. Because of the asymmetric property between VB and fermion pair, the SVBS spin chain is not self-dual at the point $|r| = 1$ and the order parameter [Eq. (44)] takes its maximum value

$$|\Delta_{\max}| = (\sqrt{5} - 2)\sqrt{\frac{2M(1+\sqrt{5})}{M+1}} \tag{96}$$

at

$$|r| = \left(M + \frac{1}{2}\right)\sqrt{\frac{1+\sqrt{5}}{2M(M+1)}}. \tag{97}$$

The expectation values for the boson number $n_b(i) = a_i^\dagger a_i + b_i^\dagger b_i$ and the fermion number $n_f(i) = f_i^\dagger f_i$ are calculated as

$$\langle n_b \rangle = 2M - 1 + \frac{2M+1}{\sqrt{4M(M+1)(1+|r|^2)+1}},$$

$$\langle n_f \rangle = 1 - \frac{2M+1}{\sqrt{4M(M+1)(1+|r|^2)+1}}. \tag{98}$$

As expected, with increasing $|r|$, $\langle n_b \rangle$ monotonically decreases, while $\langle n_f \rangle$ monotonically increases. The fluctuations for the boson number $\delta n_b^2 = \langle n_b^2 \rangle - \langle n_b \rangle^2$ and the fermion number $\delta n_f = \langle n_f^2 \rangle - \langle n_f \rangle^2$ are also evaluated as

$$\delta n_b^2 = \delta n_f^2 = x(1-x),$$

$$x = \frac{2M+1}{\sqrt{4M(M+1)(1+|r|^2)+1}}, \tag{99}$$

and their maximum is $\delta n_b = \delta n_f = \frac{1}{2}$ at $x = \frac{1}{2}$ or

$$|r| = 3\left[1 + \frac{1}{4M(M+1)}\right]. \tag{100}$$

V. HAMILTONIANS FOR THE SVBS STATE

In Secs. III and IV, we have studied the properties of the SVBS state [Eq. (4)] and its relation to the Abelian and non-Abelian fractional quantum Hall wave functions. To obtain a better understanding of what physical systems the SVBS states describe, we shall in this section construct a Hamiltonian for which the SVBS state is a unique ground state.

A. Generic truncated pseudopotential Hamiltonian

As mentioned in Sec. II, the SVBS state [Eq. (4)] is invariant under $\mathsf{OSp}(1|2)$ transformations generated by the

parameter-dependent generators, L_a and K_μ, when $x^2=-r$. Taking advantage of this symmetry, it is possible to construct pseudopotential Hamiltonians for the SVBS states with *arbitrary* values of the parameter r. Truncated pseudopotential Hamiltonians for the SVBS states [Eq. (4)] are constructed by following the similar methods of the original AKLT model. The superspin operator on site i, $L_i=\frac{1}{2}(a_i^\dagger a_i+b_i^\dagger b_i+f_i^\dagger f_i)$, acts the SVBS state to yield the eigenvalue $L=\frac{1}{2}zM$. The z component of the bond superspin $J_{ij}^z=L^z(i)+L^z(j)=\frac{1}{2}(a_i^\dagger a_i+a_j^\dagger a_j-b_i^\dagger b_i-b_j^\dagger b_j)$ counts the difference between the powers of a and b in the SVBS state [Eq. (4)] and the maximal value of J^z reads as $J_{\max}^z=(z-1)M=2L-M$. Since the SVBS state is invariant under the $\mathsf{OSp}(1|2)$ transformation, the maximal magnitude of bond superspin is equal to that of its z component, i.e., $J_{\max}=J_{\max}^z$. Thus, the SVBS state does not contain any $\mathsf{OSp}(1|2)$ angular-momentum components larger than $J_{\max}$ and is a zero-energy ground state of the truncated pseudopotential Hamiltonian,

$$H=\sum_{\langle ij\rangle}\sum_{J=J_{\max}+1/2}^{2L} V_J\mathbb{P}_J(ij), \tag{101}$$

where V_J are positive coefficients. $\mathbb{P}_J(ij)$ is the projection operator made by $\mathsf{OSp}(1|2)$ Casimir operators,

$$\begin{aligned}\mathbb{P}_J(ij)&=\prod_{J'\neq J}^{2L}\frac{[K_A(i)+K_A(j)]^2-J'\left(J'+\frac{1}{2}\right)}{J\left(J+\frac{1}{2}\right)-J'\left(J'+\frac{1}{2}\right)}\\&=\prod_{J'\neq J}^{2L}\frac{2K_A(i)K_A(j)+2L\left(L+\frac{1}{2}\right)-J'\left(J'+\frac{1}{2}\right)}{J\left(J+\frac{1}{2}\right)-J'\left(J'+\frac{1}{2}\right)},\end{aligned} \tag{102}$$

which projects to the two-site subspace of the bond superspin J. Here, we have used $K_A^2(i)=K_A^2(j)=L(L+\frac{1}{2})$ with $K_A^2=L_a^2+\epsilon_{\mu\nu}K_\mu K_\nu$. Apparently, the projection operator [Eq. (102)] is $\mathsf{OSp}(1|2)$ invariant and hence the truncated pseudopotential Hamiltonian (101) as well. Following similar discussions in the AKLT model, one may prove that the SVBS state is the unique zero-energy eigenstate of the Hamiltonian (101).

As an explicit example, it would be worthwhile to demonstrate the truncated pseudopotential Hamiltonian for the $L=1$ SVBS spin chain. With the $\mathsf{OSp}(1|2)$ decomposition rule (12), Eq. (101) becomes

$$\begin{aligned}H_{\text{chain}}&=\sum_i[V_{3/2}\mathbb{P}_{3/2}(i,i+1)+V_2\mathbb{P}_2(i,i+1)]\\&=\sum_i\left\{\frac{32}{315}(V_2-7V_{3/2})[K_A(i)K_A(i+1)]^4\right.\\&\quad+\frac{16}{45}(V_2-5V_{3/2})[K_A(i)K_A(i+1)]^3\\&\quad+\frac{2}{45}(9V_2-7V_{3/2})[K_A(i)K_A(i+1)]^2\\&\quad\left.+\frac{1}{35}(5V_2+63V_{3/2})K_A(i)K_A(i+1)+V_{3/2}\right\}.\end{aligned} \tag{103}$$

In the special case $V_2=7V_{3/2}$, the first term on the last RHS in Eq. (103) vanishes and Eq. (103) is reduced to

$$H_{\text{chain}}\to\frac{4}{45}\sum_i V_{3/2}\mathbb{P}_{3/2\oplus 2}(i,i+1), \tag{104}$$

where $\mathbb{P}_{3/2\oplus 2}$ is the projection operator onto the space with bond superspin $\frac{3}{2}$ or 2,

$$\begin{aligned}\mathbb{P}_{3/2\oplus 2}(i,i+1)&=\prod_{J=0,1/2,1}\left\{[K_A(i)+K_A(i+1)]^2-J\left[J+\frac{1}{2}\right]\right\}\\&=8[K_A(i)K_A(i+1)]^3+28[K_A(i)K_A(i+1)]^2\\&\quad+\frac{63}{2}K_A(i)K_A(i+1)+\frac{45}{4}.\end{aligned} \tag{105}$$

However, Hamiltonian (101) cannot correspond to that of any physical system since it is non-Hermitian[46] because of the term $\epsilon_{\mu\nu}K_\mu K_\nu$, as mentioned in Sec. II. To obtain a physical Hamiltonian for which the SVBS state is its unique ground state, one can replace the Hamiltonian (101) by the following form:

$$H=\sum_{\langle ij\rangle}\sum_{J=J_{\max}+1/2}^{2L} V_J\mathbb{P}_J^\dagger(ij)\mathbb{P}_J(ij), \tag{106}$$

in which $V_J>0$ just like in Eq. (101). Here we would like to make several comments on some properties of the Hermitian Hamiltonian. First, the definition (106) is a natural generalization of the original pseudopotential Hamiltonian since, if the projection operators were Hermitian, with the property $\mathbb{P}_J^2=\mathbb{P}_J$, Eq. (106) would be reduced to the original form (101). Second, unlike the non-Hermitian Hamiltonian (101), Eq. (106) is *not* $\mathsf{OSp}(1|2)$ SUSY invariant because the Hermitian conjugate of the $\mathsf{OSp}(1|2)$ Casimir operator contained in $\mathbb{P}_J^\dagger$ is no longer invariant under the original $\mathsf{OSp}(1|2)$ transformation. Consequently, the excitation spectrum of the Hermitian Hamiltonian is not SUSY invariant, even though the ground state remains is a SUSY singlet. Third, Hamiltonian (106) does not preserve the total fermion number $N_f=\Sigma_i f_i^\dagger f_i$ since the Casimir operator $[K_A(i)+K_A(j)]^2$ contains pair-creation terms of fermions, as shown in Appendix B. This is in agreement with the fermion number fluctuation in the SVBS state [Eq. (4)]. Physically, such a pseudopotential Hamiltonian describes some interacting electron system coupled with a superconducting bath, which provide a particle bath through proximity effect.

Since $\mathbb{P}_J^\dagger(ij)\mathbb{P}_J(ij)$ is always non-negative, it is straightforward to prove that $H|G\rangle=0$ for a state $|G\rangle$ if and only if $\mathbb{P}_J(ij)=0$ for all sites and all $J_{\max}<J\leq 2L$. Consequently, if the SVBS state is the only zero-energy eigenstate of Hamiltonian (101), it must also be the unique ground state of the Hermitian Hamiltonian (106). One can then prove the SVBS state to be the unique ground state of Hamiltonian (106) following exactly the same procedure as AKLT's original

work.[6,7] We will leave the detail of this proof as the task of Appendix C. Here, we sketch the proof for $L=1$ superspin chain. Let $|\Psi_G\rangle$ be a ground state of Hamiltonian (106) and satisfy the equation

$$H|\Psi_G\rangle = 0. \tag{107}$$

Then,

$$\langle\Psi_G|H|\Psi_G\rangle = 0 \Rightarrow \langle\Psi_G|\mathbb{P}_J^\dagger(ij)\mathbb{P}_J(ij)|\Psi_G\rangle_{J>J_{\max}} = 0, \tag{108a}$$

$$\Rightarrow \mathbb{P}_J(ij)|\Psi_G\rangle_{J>J_{\max}} = 0, \tag{108b}$$

where in the second arrow [Eq. (108a)] we have used that V_J in Eq. (101) satisfy $V_J>0$. Meanwhile, if $|\Psi_G\rangle$ is annihilated by the projection operator, i.e., if

$$\mathbb{P}_J(ij)|\Psi_G\rangle_{J>J_{\max}} = 0, \tag{109}$$

then it immediately follows that $H|\Psi_G\rangle=0$. Thus, the condition (109) is the necessary and sufficient condition such that the $|\Psi_G\rangle$ is the ground state of Hamiltonian (106). We use the condition (109) to show $|\Psi_G\rangle$ is the unique ground state of the Hamiltonian. For $L=1$, the condition (109) is given by

$$\mathbb{P}_{3/2}(i,i+1)|\Psi_G\rangle = \mathbb{P}_2(i,i+1)|\Psi_G\rangle = 0. \tag{110}$$

As we assumed, there is superspin 1 on each site of the chain and therefore, if the two superspins on sites i and $i+1$ did not combine a $\mathsf{OSp}(1|2)$ singlet, their bond superspin inevitably would exceed $J_{\max}=1$ due to the $\mathsf{OSp}(1|2)$ decomposition rule [Eq. (12)]. This observation holds for bond superspins on arbitrary two neighboring sites. Then, on any two neighboring sites, the bond superspin should form a $\mathsf{OSp}(1|2)$ singlet and the "bulk" ground state is given by the products of neighboring $\mathsf{OSp}(1|2)$ singlet states. Hence, with periodic boundary, it is apparent that the SVBS chain state [Eq. (16)] is the unique ground state. With open boundaries, there are ninefold quasidegenerate ground states corresponding to directions of the superspins on two ends,

$$|\Psi_G\rangle_{\mu\nu} = \psi_{\mu,0}^\dagger \cdot \prod_{i=1}^{L-1} (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger + r f_i^\dagger f_{i+1}^\dagger) \cdot \psi_{\nu,L}^\dagger |0\rangle, \tag{111}$$

where $\mu,\nu=a,b,f$. These ninefold quasidegenerate states generally take different expectation values for local observable A,

$$\langle A\rangle_{\mu\nu} = \frac{\langle\Psi_G|A|\Psi_G\rangle_{\mu\nu}}{\langle\Psi_G|\Psi_G\rangle_{\mu\nu}}. \tag{112}$$

However, as in the original AKLT case,[7] the different energy eigenvalues converge in the infinite chain limit as we shall see below. Suppose the length of the chain N (from site 0 to site N), and A takes its support in $\{l,\ldots,N-l\}$ $(l\ll N)$. First, we discuss the integration of the numerator of Eq. (112) from one end (site 0) to site l. The inner products of the superspin states at site 0 are denoted as

$$\alpha_0 + \beta_0\hat{n}_0^z + \gamma_0\theta_0\theta_0^*. \tag{113}$$

The self-inner products of u_0, v_0, and θ_0 correspond to $(\alpha_0,\beta_0,\gamma_0)=(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},-\frac{1}{2},0)$, and $(0,0,1)$, respectively. The integration from site j to site $j+1$ induces the transformation

$$\begin{pmatrix}\alpha_j\\ \beta_j\\ \gamma_j\end{pmatrix} \to \begin{pmatrix}\alpha_{j+1}\\ \beta_{j+1}\\ \gamma_{j+1}\end{pmatrix} = \begin{pmatrix}\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & -\frac{1}{2} & 0\\ |r|^2 & 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_j\\ \beta_j\\ \gamma_j\end{pmatrix}. \tag{114}$$

The three eigenvalues of the transfer matrix are given by $\lambda_\pm=(3\pm\sqrt{9+8|r|^2})/4$ and $-\frac{1}{2}$, and then, at $l\to\infty$, the product of the transfer matrices provides

$$T^l \to \frac{\lambda_+^l}{\lambda_+-\lambda_-}\begin{pmatrix}\lambda_+ & 0 & \lambda_-\\ 0 & 0 & 0\\ -\lambda_+ & 0 & -\lambda_-\end{pmatrix}. \tag{115}$$

Then, if there is u_0 or v_0 at site 0, we have a factor $(1-\theta_l\theta_l^*)\lambda_+^{l+1}/2(\lambda_+-\lambda_-)$ at site l, while if θ_0, we have a different value $(1-\theta_l\theta_l^*)\lambda_+^l\lambda_-/(\lambda_+-\lambda_-)$ but the results only differ by the scaling factor and such difference is not relevant to $\langle A\rangle_{\mu\nu}$ since the scaling factor is canceled between the numerator and the denominator in Eq. (112). Thus, the integration is not relevant to directions of the superspin at site 0 in the infinite limit. The integration from the other end (site N) to site $N-l$ gives same consequence. Then, regardless of directions of superspins on boundaries, the expectation value of any local observable provides a unique value

$$\langle A\rangle_{\mu\nu} \to \langle A\rangle, \tag{116}$$

and, in this sense, the ninefold quasidegenerate SUSY ground states converge to the unique ground state on infinite chain.

B. Another Hamiltonian for fixed total fermion number

In this subsection, we will show an alternative Hamiltonian for the simplest $L=1$ case, which is not constructed from the OSp(1|2) Casimir operators but has the advantage of respecting fermion number conservation. Motivated by the three-site Hamiltonian known for Majumdar-Ghosh spin chain,[21] here we construct a Hamiltonian with both two-site and three-site terms, for which the projection of the SVBS state [Eq. (4)] to a fixed total fermion number is a unique ground state. Such AKLT states with fixed fermion number have appeared in each order of the expansion of the SVBS state as seen in Sec. IV B. For simplicity, we will focus on the $M=1$ case, i.e., a chain with $S=1$ or $S=\frac{1}{2}$ on each site. We will first write down the form of the Hamiltonian before analyzing the physical meaning of each term.

$$H = H_t + H_V + H_U - \mu\sum_i f_i^\dagger f_i,$$

$$H_V = \sum_i [V_{3/2}P_{3/2}(i,i+1) + V_2P_2(i,i+1)],$$

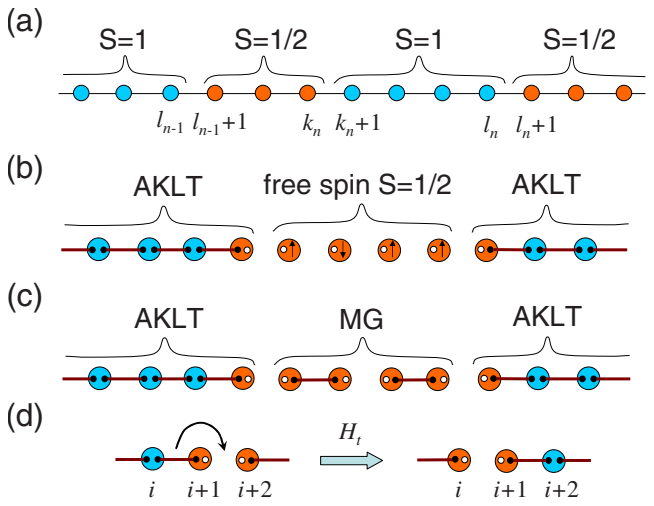

FIG. 8. (Color online) (a) Schematic picture of a spin configuration of the SVBS chain. The blue or orange sites are spin-1 and spin-$\frac{1}{2}$, respectively. k_n and l_n label the last site of each $S=\frac{1}{2}$ ($S=1$) segment. (b) Schematic picture of the ground states of H_V in Eq. (117). Each solid line stands for a nearest-neighbor singlet pair. The spin of $S=\frac{1}{2}$ sites are free except for the neighbor sites of the $S=1$ segments. (c) Schematic picture of the ground states of H_V+H_U. The $S=1$ sites form AKLT state (terminated by a $S=\frac{1}{2}$ site) and the $S=\frac{1}{2}$ sites form dimerized MG state. For a fixed configuration of $S=1$ and $S=\frac{1}{2}$ sites, the ground state is unique. (d) The effect of the hopping term H_t, which hops a nearest-neighbor singlet from i, $i+1$ link to $i+1$, $i+2$ link, or vice versa.

$$H_U = \sum_i U_{3/2} P_{3/2}(i,i+1,i+2),$$

$$H_t = -t \sum_i (\Delta_{i,i+1}\Delta^\dagger_{i+1,i+2} + \text{H.c.}) \tag{117}$$

with $P_J(i,i+1)$ and $P_J(i,i+1,i+2)$ the two-site and three-site projections to total SU(2) spin J states, respectively, and $\Delta_{i,i+1}=f_i^\dagger f_{i+1}^\dagger(a_i b_{i+1}-b_i a_{i+1})$ the annihilation operator of a Cooper pair. It should be noticed that the Hamiltonian is defined in the Hilbert space satisfying the constraint $a_i^\dagger a_i + b_i^\dagger b_i + f_i^\dagger f_i = 2, \forall i$. The coefficients V_2, $U_{\frac{3}{2}}$, t are all positive. The chemical-potential term $-\mu\Sigma_i f_i^\dagger f_i$ determines the fermion number in the ground states.

To understand the ground-state property of Hamiltonian (117), we start from the interaction terms H_V+H_U. Since H_V+H_U preserves the fermion number $n_i^h=f_i^\dagger f_i$ on each lattice site, one can focus on studying its matrix element within a subspace defined by fixed eigenvalue of n_i^h. For any given configuration $\{n_i^h\}_{i=1}^N$, the 1D chain can be viewed as consecutive staggered sectors of spin-1 and spin-$\frac{1}{2}$ chains, as shown in Fig. 8(a). When $\{n_i^h\}$ satisfies

$$n_i^h = \begin{cases} 0 & \text{for } k_n < i \le l_n \\ 1 & \text{for } l_n < i \le k_{n+1} \end{cases}$$

with $n \in \{1,\ldots,M\}$, the chain consists of M spin-1 chains with lengths l_n-k_n and M spin-$\frac{1}{2}$ chains with lengths $k_{n+1}-l_n$. (Here $k_{M+1}=k_1$.)

Now we consider the effect of H_V and H_U on such a spin chain. First, the two-site projector $P_2(i,i+1)$ is nontrivial only when there are no fermion on the two sites $(i,i+1)$ because the total spin is automatically smaller than 2 if there are one or two holes on these two sites. Therefore, the V_2 term in H_V is an AKLT Hamiltonian acting on the disconnected spin-1 segments $k_n<i\le l_n$. Thus we immediately know that the V_2 term takes the minimal eigenvalue of zero if the spin-1 segments $k_n<i\le l_n$ are all spin-1 AKLT spin chains.

Second, the two-site projector $P_{3/2}(i,i+1)$ is nontrivial only when there is one fermion on the two sites $(i,i+1)$, i.e., $n_i^h+n_{i+1}^h=1$. For these sites, the requirement $P_{3/2}(i,i+1)=0$ leads to singlet pair between the free $S=\frac{1}{2}$ spin at the end of the AKLT spin-1 chain and the neighbor spin-$\frac{1}{2}$ site. This requirement automatically fixes the length of each spin-$\frac{1}{2}$ segment $k_{n+1}-l_n$ to be ≥ 2. Other spin-$\frac{1}{2}$ sites which are not neighbor of spin-1 site are not affected by H_V. In summary, the spin configuration with vanishing eigenvalue of H_V is shown in Fig. 8(b).

Third, the three-site projector $P_{3/2}(i,i+1,i+2)$ is nontrivial only when there are one or three fermions on the three sites $(i,i+1,i+2)$. When there are one fermion on the three sites, it can be proved that any spin configuration which satisfy $H_V=0$ also satisfy $H_U=0$ automatically. Thus we only need to consider the effect of H_U on the sites with three fermions, i.e., three consecutive sites with $n_i^h=n_{i+1}^h=n_{i+2}^h=1$. In other words, H_U is exactly the Majumdar-Ghosh Hamiltonian for the $S=\frac{1}{2}$ segments. As known from the work of Majumdar and Ghosh, the ground-state requirement $H_U=0$ can only be satisfied by the two valence bond solid states, with spin singlet pairs between each two nearest-neighbor sites. Moreover, the connect condition to the $S=1$ segments will pick one of the two VBS states, as shown in Fig. 8(c). (Also, the length of each $S=\frac{1}{2}$ segment is automatically required to be even, in order to form singlet pairs.)

In summary, the ground state of interaction terms H_U+H_V is uniquely determined for a given distribution of $S=1$ and $S=\frac{1}{2}$ sites. Now we consider the effect of the hopping term H_t. The operator $\Delta_{i,i+1}$ annihilates a singlet pair and creates two fermions on i and $i+1$ sites. Thus $\Delta_{i,i+1}\Delta^\dagger_{i+1,i+2}$ flips a singlet from $i,i+1$ link to $i+1$, $i+2$ link. Notice that $\Delta^\dagger_{i+1,i+2}=(a^\dagger_{i+1}b^\dagger_{i+2}-b^\dagger_{i+1}a^\dagger_{i+2})f_{i+2}f_{i+1}$, we know that the term $\Delta_{i,i+1}\Delta^\dagger_{i+1,i+2}$ has nonzero matrix element only if $n^h_{i+1}=n^h_{i+2}=1$, $n^h_i=0$. In other words, H_t only acts on the interface sites between $S=1$ and $S=\frac{1}{2}$ segments. Moreover, in the ground-state manifold of H_V+H_U, the effect of H_t is simply hopping of a nearest-neighbor singlet, as shown in Fig. 8(d). From this picture we know that H_t preserves a ground state of H_V+H_U in the ground-state manifold. Consequently, H_t lifts the degeneracy of the ground-state manifold of H_V+H_U. The lowest energy state determined by H_t for a fixed total fermion number is obviously the equal weight superposition of all the spin configurations satisfying $H_V+H_U=0$, which is exactly the SVBS state [Eq. (4)] projected to a fixed fermion number,

$$|G_N\rangle = P_N \prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger + f_i^\dagger f_j^\dagger)|0\rangle. \tag{118}$$

It should be noticed that $|G_N\rangle$ is nonvanishing only when N is even, otherwise the ground state cannot be a spin singlet. As

the last step, the fermion number N for which the state $|G_N\rangle$ has lowest energy can be tuned by the chemical-potential term $-\mu\Sigma_i f_i^\dagger f_i$. It is possible that for some μ the ground state contains odd number of fermions, which thus cannot be SVBS state.

In conclusion, we have shown that Hamiltonian (117) has the SVBS state [Eq. (118)] as its unique ground state, as long as t, $V_{3/2}$, V_2, $U_{3/2}>0$ and the chemical potential is chosen properly so that the ground state has even number of fermions. We have also confirmed this fact numerically by diagonalizing Hamiltonian (117) for up to five sites with periodic boundary condition and calculating the overlap between the numerical ground-state wave function and the projected SVBS state [Eq. (118)]. Within numerical accuracy, the ground state of Hamiltonian (117) for even total fermion number is unique and always given by the SVBS state [Eq. (118)].

VI. CONCLUSIONS

In conclusion we have constructed the supersymmetric generalization of the valence bond solid states. In one dimension, these SVBS states smoothly interpolate between the integer and half-integer VBS states, and they represent superconducting valence bond liquid states. We also constructed microscopic Hamiltonians for which these states are the exact quantum ground states. We show that the SVBS states are analogous to bosonic Pfaffian states of the quantum Hall effect, in precisely the same sense as the analogy between the VBS states and the Laughlin quantum Hall states. Our work also provides a precise mathematical realization of some ideas in strongly correlated systems, in the sense that the doped valence bond liquid states are naturally superconducting, and that the superconducting states can be obtained from a symmetry rotation, in our case a supersymmetric rotation, of the quantum antiferromagnetic ground states. For the future, we propose to focus on the two- and higher-dimensional versions of the SVBS states. Given the analogies between the SVBS states and the Pfaffian states in the quantum Hall effect, it would also be interesting to explore the possibility of non-Abelian statistics of the elementary excitations.

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APPENDIX A: OSp(1|2) AND SU(2|1) ALGEBRAS

Here, we review basic properties of $\mathsf{OSp}(1|2)$ and $\mathsf{SU}(2|1)$ algebras with emphasis on their relation to Schwinger boson and slave fermion formalism. The $\mathsf{OSp}(1|2)$ algebra consists of five generators, $L_A=L_a(a=1,2,3)$ and $L_\mu(\mu=\theta_1,\theta_2)$ that satisfy

$$[L_a,L_b]=i\epsilon_{abd}L_c,$$

$$[L_a,L_\mu]=\frac{1}{2}(\sigma_a)_{\nu\mu}L_\nu,$$

$$\{L_\mu,L_\nu\}=\frac{1}{2}(\epsilon\sigma_a)_{\mu\nu}L_a, \tag{A1}$$

where σ_a are Pauli matrices and ϵ is the 2×2 antisymmetric matrix $\epsilon=i\sigma_2$. Equation (A1) suggests that L_a transform as $\mathsf{SU}(2)$ vector and L_μ $\mathsf{SU}(2)$ spinor. The Casimir operator for the $\mathsf{OSp}(1|2)$ group is given by

$$C=L_AL_A\equiv L_aL_a+\epsilon_{\mu\nu}L_\mu L_\nu, \tag{A2}$$

and its eigenvalue is $L(L+\frac{1}{2})$ with integer of half-integer L. L is referred to as superspin and characterizes the irreducible representations of $\mathsf{OSp}(1|2)$. The dimension of irreducible representation with superspin L is $4L+1$, $2L+1$ of which is the $\mathsf{SU}(2)$ spin L representation, and the remaining $2L$ is $\mathsf{SU}(2)$ spin $L-\frac{1}{2}$. Specifically, the $\mathsf{OSp}(1|2)$ fundamental representation $L=\frac{1}{2}$ is three-component spinor and the corresponding $\mathsf{OSp}(1|2)$ generators are the following 3×3 matrices:

$$l_a=\frac{1}{2}\begin{pmatrix}\sigma_a & 0\\ 0 & 0\end{pmatrix},\quad l_\mu=\frac{1}{2}\begin{pmatrix}0 & \tau_\mu\\ -(\epsilon\tau_\mu)^t & 0\end{pmatrix}, \tag{A3}$$

with $\tau_1=(1,0)^t$ and $\tau_2=(0,1)^t$. The irreducible decomposition for superspin representations is given by

$$L\otimes L'=|L-L'|\oplus|L-L'|+\frac{1}{2}\oplus|L-L'|+1\oplus\cdots\oplus L+L'. \tag{A4}$$

Unlike the $\mathsf{SU}(2)$ decomposition rule, the superspins on the RHS differ by $\frac{1}{2}$.

The $\mathsf{SU}(2|1)$ or $\mathsf{OSp}(2|2)$ algebra consists of eight generators; L_a, L_μ [$\mathsf{OSp}(1|2)$ generators], D_μ and Γ that satisfy

$$[L_a,D_\mu]=\frac{1}{2}(\sigma_a)_{\nu\mu}D_\nu,$$

$$\{D_\mu,D_\nu\}=-\frac{1}{2}(\epsilon\sigma_a)_{\mu\nu}L_a,$$

$$\{L_\mu,D_\nu\}=-\frac{1}{4}\epsilon_{\mu\nu}\Gamma,$$

$$[L_a,\Gamma]=0,$$

$$[L_\mu,\Gamma]=-D_\mu,$$

$$[D_\mu,\Gamma]=-L_\mu. \tag{A5}$$

With Eq. (A3), the simplest matrix realization for Eq. (A5) is given by

$$d_\mu = \frac{1}{2}\begin{bmatrix} 0 & -\tau_\mu \\ -(\epsilon\tau_\mu)^t & 0 \end{bmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{A6}$$

As the Schwinger particle used in the SU(2) spin formalism, slave fermion is introduced in the superspin formalism. We denote Schwinger bosons as SU(2) spinor $b_\mu=(a,b)$ and slave fermion as SU(2) singlet f, and they satisfy the commutation relations: $[a,a^\dagger]=[b,b^\dagger]=\{f,f^\dagger\}=1$. The SU(2|1) operators are represented as

$$L_a = \psi^\dagger l_a \psi = \frac{1}{2}(\sigma_a)_{\mu\nu} b_\mu^\dagger b_\nu,$$

$$L_\mu = \psi^\dagger l_\mu \psi = \frac{1}{2}(b_\mu^\dagger f + \epsilon_{\mu\nu} f^\dagger b_\nu),$$

$$D_\mu = \psi^\dagger d_\mu \psi = \frac{1}{2}(-b_\mu^\dagger f + \epsilon_{\mu\nu} f^\dagger b_\nu),$$

$$\Gamma = \psi^\dagger \gamma \psi = a^\dagger a + b^\dagger b + 2f^\dagger f, \tag{A7}$$

where $\psi=(a,b,f)^t=(b_1,b_2,f)^t$. L_μ and D_μ are not Hermitian in the conventional sense, while with the definition of the superstar conjugation ‡

$$(f^\ddagger)^\ddagger = -f, \quad (f_1 f_2)^\ddagger = f_1^\ddagger f_2^\ddagger, \tag{A8}$$

they become pseudoHermitian operators

$$L_\mu^\ddagger = \epsilon_{\mu\nu} L_\nu, \quad D_\mu^\ddagger = -\epsilon_{\mu\nu} D_\nu \tag{A9}$$

(The detail definition of the superstar conjugation can be referred to Ref. 47). In the slave fermion representation, the OSp(1|2) Casimir operator (A2) is rephrased as

$$C = \frac{a^\dagger a + b^\dagger b + f^\dagger f}{2}\left(\frac{a^\dagger a + b^\dagger b + f^\dagger f}{2} + \frac{1}{2}\right), \tag{A10}$$

and the superspin magnitude corresponds to the half of the total particle number

$$L = \frac{1}{2}(a^\dagger a + b^\dagger b + f^\dagger f). \tag{A11}$$

We introduce a complex parameter x to define one-parameter family of fermionic generators made by L_μ and D_μ

$$K_\mu = \frac{1}{2x}(L_\mu - D_\mu) + \frac{x}{2}(L_\mu + D_\mu),$$

$$= \frac{1}{2}\begin{pmatrix} a \\ b \\ f \end{pmatrix}^\dagger \begin{pmatrix} 0 & \frac{1}{x}\tau_\mu \\ -x(\epsilon\tau_\mu)^t & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ f \end{pmatrix}$$

$$= \frac{1}{2x} b_\mu^\dagger f + \frac{x}{2}\epsilon_{\mu\nu} f^\dagger b_\nu. \tag{A12}$$

At $x=1$, K_μ is reduced to L_μ and at $x=i$, $K_\mu=iD_\mu$. Though K_μ depends on the parameter x, interestingly, L_a and K_μ satisfy the parameter-*independent* OSp(1|2) algebraic relations

$$[L_a, L_b] = i\epsilon_{abc} L_c,$$

$$[L_a, K_\mu] = \frac{1}{2}(\sigma_a)_{\nu\mu} K_\nu,$$

$$\{K_\mu, K_\nu\} = \frac{1}{2}(\epsilon\sigma_a)_{\mu\nu} L_a. \tag{A13}$$

The Casimir operator is given by

$$K_A^2 \equiv L_a^2 + \epsilon_{\mu\nu} K_\mu K_\nu = L_a^2 + \left(\frac{x}{2} + \frac{1}{2x}\right)^2 \epsilon_{\mu\nu} L_\mu L_\nu + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 \epsilon_{\mu\nu} D_\mu D_\nu, \tag{A14}$$

which, in the slave fermion representation, is expressed as

$$K_A^2 = \frac{a^\dagger a + b^\dagger b + f^\dagger f}{2}\left(\frac{a^\dagger a + b^\dagger b + f^\dagger f}{2} + \frac{1}{2}\right). \tag{A15}$$

Again, the parameter x does not appear in Eq. (A15) and the eigenvalues of the Casimir operator are given by $L(L+1/2)$ for any of the one-parameter family.

APPENDIX B: SUSY SPIN-SPIN INTERACTIONS

Here, we discuss several properties of the OSp(1|2) spin-spin interaction

$$K_A(i)K_A(j) = L_a(i)L_a(j) + \epsilon_{\mu\nu} K_\mu(i) K_\nu(j), \tag{B1}$$

where i and j represent the sites on which superspins are defined. Since the SUSY spin-spin interaction operator commutes with the superspin-magnitude operator $L_i=1/2(a^\dagger a + b^\dagger b + f^\dagger f)_i$, the SUSY spin-spin interaction does not change the magnitude of the superspin on each site. The bosonic spin-spin interaction part of Eq. (B1) gives the SU(2) spin-spin interaction

$$L_a(i)L_a(j) = \frac{1}{2} a_i^\dagger a_j b_j^\dagger b_i + \frac{1}{2} a_j^\dagger a_i b_i^\dagger b_j + \frac{1}{4}(a^\dagger a - b^\dagger b)_i (a^\dagger a - b^\dagger b)_j, \tag{B2}$$

while the fermionic spin-spin interaction part of Eq. (B1) provides

$$\epsilon_{\mu\nu} K_\mu(i) K_\nu(j) = \left(\frac{x}{2} + \frac{1}{2x}\right)^2 \epsilon_{\mu\nu} L_\mu(i) L_\nu(j) + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 \epsilon_{\mu\nu} D_\mu(i) D_\nu(j) + \left(\frac{x}{2} + \frac{1}{2x}\right)\left(\frac{x}{2} - \frac{1}{2x}\right)\epsilon_{\mu\nu}[L_\mu(i) D_\nu(j) + D_\mu(i) L_\nu(j)], \tag{B3}$$

and, in the Slave fermion representation, expressed as

$$\epsilon_{\mu\nu}K_\mu(i)K_\nu(j) = \frac{1}{4x^2}(a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)f_i f_j + \frac{x^2}{4}(a_i b_j - b_i a_j)f_i^\dagger f_j^\dagger + \frac{1}{4}(a_i^\dagger a_j + b_i^\dagger b_j)f_j^\dagger f_i + \frac{1}{4}(a_j^\dagger a_i + b_j^\dagger b_i)f_i^\dagger f_j. \tag{B4}$$

The first two terms on the RHS in Eq. (B4) are particular interactions existing in the $\mathsf{OSp}(1|2)$ spin-spin term. They violate the total fermion number conservation and represents hole-pair annihilating and creating interactions. Besides, they are not Hermitian even at $x=1$. [However, at $x=1$, they are pseudo-Hermitian with the definition of the superstar conjugation (A8).] The last two terms represent interchange of fermion and boson between i and j sites. Meanwhile, the $\mathsf{SU}(2|1)$ spin-spin interaction is given by

$$\begin{aligned} &L_a(i)L_a(j) + \epsilon_{\mu\nu}L_\mu(i)L_\nu(j) - \epsilon_{\mu\nu}D_\mu(i)D_\nu(j) - \frac{1}{4}\Gamma(i)\Gamma(j) \\ &= \frac{1}{2}a_i^\dagger a_j b_j^\dagger b_i + \frac{1}{2}a_j^\dagger a_i b_i^\dagger b_j + \frac{1}{4}(a^\dagger a - b^\dagger b)_i(a^\dagger a - b^\dagger b)_j \\ &\quad + \frac{1}{2}(a_j^\dagger a_i + b_j^\dagger b_i)f_i^\dagger f_j + \frac{1}{2}(a_i^\dagger a_j + b_i^\dagger b_j)f_j^\dagger f_i \\ &\quad - \frac{1}{4}(a^\dagger a + b^\dagger b + 2f^\dagger f)_i(a^\dagger a + b^\dagger b + 2f^\dagger f)_j, \end{aligned} \tag{B5}$$

and is the component of the SUSY $t-J$ model Hamiltonian. It should be noted that the particular hole-pair creating and annihilating terms in Eq. (B4) do not exist in the $\mathsf{SU}(2|1)$ spin-spin interaction.

Though the $\mathsf{OSp}(1|2)$-invariant spin-spin interaction is not Hermitian, its eigenvalues are real and do not depend on the parameter x. Indeed, with two-body operator $K_A(i,j) = K_A(i) + K_A(j)$, the $\mathsf{OSp}(1|2)$ spin-spin interaction (B1) is simply rewritten as

$$K_A(i)K_A(j) = \frac{1}{2}K_A(i,j)^2 - \frac{1}{2}K_A(i)^2 - \frac{1}{2}K_A(j)^2, \tag{B6}$$

and its eigenvalues are

$$E = \frac{1}{2}J\left(J+\frac{1}{2}\right) - \frac{1}{2}L_i\left(L_i+\frac{1}{2}\right) - \frac{1}{2}L_j\left(L_j+\frac{1}{2}\right), \tag{B7}$$

where J, L_i, and L_j are the Casimir indexes for $K_A(i,j)$, $K_A(i)$, and $K_A(j)$, respectively.

One may confirm above features with a low-energy example. The two-body states $|J,J_3\rangle$ made by $L_i=\frac{1}{2}$ and $L_j=\frac{1}{2}$, carry the $\mathsf{OSp}(1|2)$ Casimir indexes $J=0\frac{1}{2},1$ by the decomposition rule (A4). The $J=0$ sector consists of

$$|0,0\rangle = (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger - x^2 f_i^\dagger f_j^\dagger)|0\rangle. \tag{B8}$$

This is the $\mathsf{OSp}(1|2)$ singlet state and is the "component" of the SVBS state [Eq. (4)]. The $J=\frac{1}{2}$ sector consists of

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = (a_i^\dagger f_j^\dagger - f_i^\dagger a_j^\dagger)|0\rangle,$$

$$\left|\frac{1}{2},0\right\rangle = (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger - 2x^2 f_i^\dagger f_j^\dagger)|0\rangle,$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = (b_i^\dagger f_j^\dagger - f_i^\dagger b_j^\dagger)|0\rangle. \tag{B9}$$

Similarly, the $J=1$ sector consists of

$$|1,1\rangle = a_i^\dagger a_j^\dagger|0\rangle,$$

$$\left|1,\frac{1}{2}\right\rangle = (a_i^\dagger f_j^\dagger + f_i^\dagger a_j^\dagger)|0\rangle,$$

$$|1,0\rangle = (a_i^\dagger b_j^\dagger + b_i^\dagger a_j^\dagger)|0\rangle,$$

$$\left|1,-\frac{1}{2}\right\rangle = (b_i^\dagger f_j^\dagger + f_i^\dagger b_j^\dagger)|0\rangle,$$

$$|1,-1\rangle = b_i^\dagger b_j^\dagger|0\rangle. \tag{B10}$$

Equation (B7) suggests that $J=0$, $J=\frac{1}{2}$, and $J=1$ sectors carry eigenvalues $E=-\frac{1}{2},-\frac{1}{4}$, and $\frac{1}{4}$, respectively. By applying the $\mathsf{OSp}(1|2)$-invariant spin-spin interaction operator to these states, one may confirm such parameter-independent eigenvalues are obtained. The parameter dependence appears only in the eigenstates $|0,0\rangle$ and $|\frac{1}{2},0\rangle$, as found in Eqs. (B8) and (B9).

APPENDIX C: PROOF OF THE SVBS STATE AS UNIQUE GROUND STATE OF HAMILTONIAN (106)

In this appendix we will prove that the SVBS state [Eq. (4)] is unique ground state of Hamiltonian (106). The procedure of this proof is a straightforward supersymmetric generalization of AKLT's original work.[6,7] To finish the proof, we need to consider the open-boundary condition. The boson and fermion can be written in a $\mathsf{OSp}(1|2)$ spinor $\psi_i = (a_i, b_i, f_i)$ and the SVBS state can be written as

$$|\mathrm{SVBS}\rangle = \prod_i (\psi_{i\mu}^\dagger C^{\mu\nu}\psi_{i+1,\nu}^\dagger)^M|0\rangle, \tag{C1}$$

where

$$C^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r \end{pmatrix}. \tag{C2}$$

For a open-boundary chain with length L, the definition needs to be modified by

$$|\mathrm{SVBS};\{\mu_s,\nu_t\}\rangle = \left(\prod_{s=1}^M \psi_{1\mu_s}^\dagger\right)\prod_{i=1}^{L-1}(\psi_{i\sigma}^\dagger C^{\sigma\tau}\psi_{i+1,\tau}^\dagger)^M\left(\prod_{t=1}^M \psi_{L\nu_t}^\dagger\right) \times|0\rangle \equiv \hat{\Omega}_{\mu_s\nu_t}|0\rangle \tag{C3}$$

in which $\hat{\Omega}_{\mu_s\nu_t} \equiv \hat{\Omega}_{\mu_1\mu_2\cdots\mu_M;\nu_1\nu_2\cdots\nu_M}$ is symmetric under the permutations $(\mu_1\mu_2\cdots\mu_M)$ and $(\nu_1\nu_2\cdots\nu_M)$. In other words, the state $|\mathrm{SVBS};\{\mu_s,\nu_t\}\rangle$ carries the $\mathsf{OSp}(1|2)$ representa-

tion $\frac{M}{2}\otimes\frac{M}{2}$. For the open-boundary system, we have the following lemma:

(i) Lemma 1. On an open-boundary chain with length L, if a state $|\Psi\rangle$ satisfies $P^N_{i,i+1}|\Psi\rangle=0,\ \forall i=1,2,\ldots,L-1,\ N=M+\frac{1}{2},\ldots,2M$, then the state is a superposition of the SVBS states (C3), i.e.,

$$|\Psi\rangle = A^{\mu_s\nu_t}|\mathrm{SVBS},\{\mu_s,\nu_t\}\rangle,\ \exists A^{\mu_s\nu_t}. \tag{C4}$$

Lemma 1 can be proved by induction as follows:

(1) In the two-site case $L=2$, the states in the Hilbert space are classified by the superspin as

$$M\otimes M = 0\oplus\frac{1}{2}\oplus 1\oplus\cdots\oplus 2M. \tag{C5}$$

The requirement $\mathbb{P}^N_{12}|\Psi\rangle=0,\ \forall N=M+\frac{1}{2},\ldots,2M$ requires the state to stay in the subHilbert space of $0\oplus\frac{1}{2}\oplus 1\oplus\cdots\oplus M$ which has a dimension of $1+3+\cdots+(4M+1)=(2M+1)^2$. On the other hand, the $(2M+1)^2$ states $|\mathrm{SVBS},\{\mu_s,\nu_t\}\rangle$ are linearly independent and satisfy the constraint. Consequently, the states $|\mathrm{SVBS},\{\mu_s,\nu_t\}\rangle$ span a complete basis of the ground-state Hilbert space. In other words, the lemma 1 for $L=2$ is proved.

(2) An arbitrary state $|\Psi\rangle_{1,L+1}$ in the Hilbert space of a length $L+1$ chain can always be expanded as $|\Psi\rangle_{1,L+1}=\Sigma_{n,m}|n\rangle_{1,L}\Psi_{nm}\otimes|m\rangle_{L+1}$ with $|n\rangle_{1,L}$ and $|m\rangle_{L+1}$ an arbitrary set of basis states for the Hilbert subspace of the first L sites and that of the last site. By an SVD decomposition of the matrix Ψ_{nm}, one can always obtain the form

$$|\Psi\rangle_{1,L+1} = \sum_k \lambda_k |W_k\rangle_{1,L}\otimes|S_k\rangle_{L+1}, \tag{C6}$$

where $|W_k\rangle_{1,L}$ are orthogonal states in the Hilbert space of a length-L chain and $|S_k\rangle$ are orthogonal states in the Hilbert space of the $L+1$th site. The coefficients $\lambda_k>0$. If $\mathbb{P}^N_{i,i+1}|\Psi\rangle_{1,L+1}=0$ for $i=1,2,\ldots,L-1$, we have

$$\begin{aligned}0 &= \mathrm{norm}\left[\mathbb{P}^N_{i,i+1}\sum_k \lambda_k|W_k\rangle_{1,L}\otimes|S_k\rangle_{L+1}\right]\Rightarrow 0\\ &= \sum_k \lambda_k^2\langle W_k|\mathbb{P}^{N\dagger}_{i,i+1}\mathbb{P}^N_{i,i+1}|W_k\rangle_{1,L}\Rightarrow 0=\mathbb{P}^N_{i,i+1}|W_k\rangle_{1,L}\Rightarrow 0\\ &= |W_k\rangle_{1,L}=A_k^{\mu_s\nu_t}|\mathrm{SVBS},\{\mu_s,\nu_t\}\rangle_{1,L}.\end{aligned} \tag{C7}$$

The last step is inductive, assuming the result holds true for a system of L sites. Thus the state $|\Psi\rangle_{1,L+1}$ is written as

$$|\Psi\rangle_{1,L+1} = \sum_k \lambda_k A_k^{\mu_s\nu_t}B_k^{\sigma_k\tau_l}\hat{\Omega}^{1,L}_{\mu_s\nu_t}\hat{\Omega}^{L+1}_{\sigma_k\tau_l}|0\rangle \tag{C8}$$

in which $\hat{\Omega}^{L+1}_{\sigma_k\tau_l}=\Pi_{k=1}^M\psi^\dagger_{L+1,\sigma_k}\Pi_{l=1}^M\psi^\dagger_{L+1,\tau_l}|0\rangle$. The indices (ν_s,σ_k) carry the representation $\frac{M}{2}\otimes\frac{M}{2}$, which can be decomposed into irreducible representations as $\frac{M}{2}\otimes\frac{M}{2}=0\oplus\frac{1}{2}\oplus\cdots\oplus M$. Such a decomposition can be expressed as

$$|\Psi\rangle_{1,L+1} = \sum_{N=0}^M\sum_{n=-N}^N F^{\mu_s\tau_l}_{Nn}C^{\nu_t\sigma_k}_{Nn}\hat{\Omega}^{1,L}_{\mu_s\nu_t}\hat{\Omega}^{L+1}_{\sigma_k\tau_l}|0\rangle \tag{C9}$$

in which $C^{\nu_t\sigma_k}_{Nn},n=-N,-N+\frac{1}{2},\ldots,N$ are the $3j$ symbols carrying the representation of $\frac{M}{2}\otimes\frac{M}{2}\otimes N$. Thus in the state $C^{\nu_t\sigma_k}_{Nn}\hat{\Omega}^{1,L}_{\mu_s\nu_t}\hat{\Omega}^{L+1}_{\sigma_k\tau_l}|0\rangle$, the sites L and $L+1$ carry the representation $\frac{M}{2}\otimes N\otimes\frac{M}{2}$. Thus we know that the maximal total $\mathsf{OSp}(1|2)$ "angular momentum" of these two sites is $M+N$. Consequently, the requirement $\mathbb{P}^N_{L,L+1}|\Psi\rangle_{1,L+1}=0,\ N>M$ requires that only $N=n=0$ terms are nonzero in Eq. (C9). In other words, the state can be written as

$$|\Psi\rangle_{1,L+1} = F^{\mu_s\tau_l}_{00}C^{\nu_t\sigma_k}_{00}\hat{\Omega}^{1,L}_{\mu_s\nu_t}\hat{\Omega}^{L+1}_{\sigma_k\tau_l}|0\rangle. \tag{C10}$$

Moreover, the coefficient $C^{\nu_t\sigma_k}_{00}$ can be expressed as

$$C^{\nu_t\sigma_k}_{00} = \mathcal{S}\left[\prod_{s=1}^M C^{\nu_s\sigma_s}\right] \tag{C11}$$

in which $\mathcal{S}[\cdots]$ stands for symmetrization over the indices $\{\nu_t\}$ and $\{\sigma_k\}$, respectively. By the definition of $\Omega^{1,L}_{\mu_s\nu_t}$ in Eq. (C3), it is straightforward to show that $C^{\nu_t\sigma_k}_{00}\hat{\Omega}^{1,L}_{\mu_s\nu_t}\hat{\Omega}^{L+1}_{\sigma_k\tau_l}=\Omega^{1,L+1}_{\mu_s\tau_l}$ so that

$$|\Psi\rangle_{1,L+1} = F^{\mu_s\tau_l}_{00}\hat{\Omega}^{1,L+1}_{\mu_s\tau_l}|0\rangle = F^{\mu_s\tau_l}_{00}|\mathrm{SVBS},\{\mu_s,\tau_l\}\rangle. \tag{C12}$$

In summary, we have proved lemma 1 by induction. By making use of lemma 1, it is straightforward to prove that the SVBS state [Eq. (4)] to be the unique ground state of Hamiltonian (106). First of all, it is easy to see that for any physical state $|\Psi\rangle$, $\langle\Psi|H|\Psi\rangle=\Sigma_i\Sigma_{N=M+1/2}^{2M}V_N\cdot\mathrm{norm}(\mathbb{P}^N_{i,i+1}|\Psi\rangle)\geq 0$. Since the SVBS state [Eq. (4)] satisfies $H|\mathrm{SVBS}\rangle=0$, we know that it is a ground state of Hamiltonian (106). On the other hand, if there is another state $|G\rangle$ satisfying $H|G\rangle=0$, we have

$$\mathrm{norm}(\mathbb{P}^N_{i,i+1}|G\rangle)=0\Rightarrow\mathbb{P}^N_{i,i+1}|G\rangle=0,\ \forall\ i,\ \forall\ M<N\leq 2M. \tag{C13}$$

Consider a chain with L sites and periodic boundary condition. According to lemma 1, the conditions $\mathbb{P}^N_{i,i+1}|G\rangle=0$ for $i=1,2,\ldots,L-1$ lead to

$$|G\rangle = A^{\mu_s\nu_t}\hat{\Omega}^{1,L}_{\mu_s\nu_t}|0\rangle.$$

In the same way as has been used in the proof of lemma 1, the coefficient $A^{\mu_s\nu_t}$ can be decomposed into different irreducible representations as

$$A^{\mu_s\nu_t} = \sum_{N=0}^M\sum_{n=-N}^N F_{Nn}C^{\mu_s\nu_t}_{Nn}. \tag{C14}$$

Applying the condition $\mathbb{P}^N_{L,1}|G\rangle=0$ to the state $|G\rangle=\Sigma_{N,n}F_{Nn}C^{\mu_s\nu_t}_{Nn}\hat{\Omega}^{1,L}_{\mu_s\nu_t}|0\rangle$ we obtain $F_{Nn}=0$ for all $N\neq 0$. Thus we have proved that

$$|G\rangle = C^{\mu_s\nu_t}_{00}\hat{\Omega}^{1,L}_{\mu_s\nu_t}|0\rangle = |\mathrm{SVBS}\rangle. \tag{C15}$$

In summary, the state $|\mathrm{SVBS}\rangle$ in Eq. (4) is the unique ground state of the generalized pseudopotential Hamiltonian (106).

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